Advanced Higher Maths

Unit 1.5

Curve Sketching

Outcome 4 – Properties of Functions

Find the vertical asymptote(s) of a rational function. (a function of the form $\dfrac{P(x)}{Q(x)}$)

Asymptotes

· An **Asymptote** is a straight line to which a curve approaches more and more closely as x becomes larger or smaller, or approaches a certain value.

Use the computer or a graphics calculator to obtain the graphs of the following rational functions.

Show the asymptotes clearly and write down their equations.

1. $f(x) = \dfrac{1}{x}$

2. $f(x) = \dfrac{1}{x-3}$

3. $f(x) = \dfrac{1}{x+2}$

4. $f(x) = \dfrac{2x+3}{3-x}$

5. $f(x) = \dfrac{1}{(x-1)(x+3)}$

6. $f(x) = \dfrac{x}{x-1}$

7. $f(x) = \dfrac{x}{(x+2)(x-3)}$

8. $f(x) = \dfrac{x^2}{(x+2)(x-3)}$

9. $f(x) = \dfrac{x-1}{x^2}$

10. $f(x) = \dfrac{x^2}{x-1}$

11. $f(x) = \dfrac{x^3}{x^2-1}$

12. $f(x) = \dfrac{1}{x^2+1}$

When using a computer or a graphics calculator, it is not always possible to be sure that you have identified the equation of the asymptote correctly.
A little algebra will identify the equations.
You will have seen that there are **vertical** and **non - vertical** asymptotes.

Vertical asymptotes are found from the zeros of the denominator ie. are in the form $x = k$
The way the curve approaches the asymptotes must also be detemined.

Examples

1. (If the degree of the numerator < the degree of the denominator) :-

e.g. $f(x) = \dfrac{2x+3}{x^2+5x+4} = \dfrac{2x+3}{(x+4)(x+1)}$

Vertical asymptotes occur at the zeros of the denominator
set $(x+4)(x+1) = 0$
=> $x = -4$ and $x = -1$ are asymptotes. (vertical lines)

We now have to determine how the curve approaches these asymptotes.
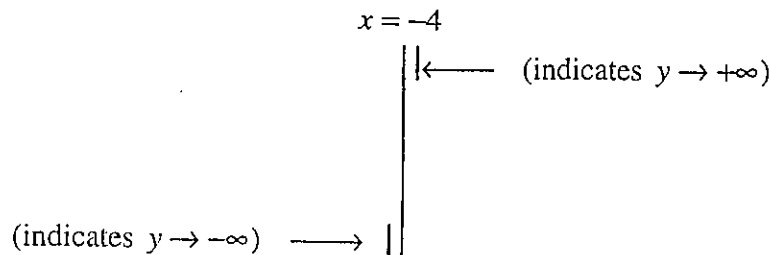As $x \to -4^-$ (means x tends to -4 from a negative direction ie. $x < -4$)

$y \to \dfrac{(-)}{(-)(-)} \to -\infty$ (assign a sign to each part of the rational function and find the net sign. , $-\infty$ means negative infinity)

cont'd ....

As $x \to -4^+$ (means x tends to -4 from a positive direction ie. $x > -4$)

$y \to \dfrac{(-)}{(+)(-)} \to +\infty$ (assign a sign to each part of the rational function and

find the <u>net</u> <u>sign</u>. $+\infty$ means positive infinity)
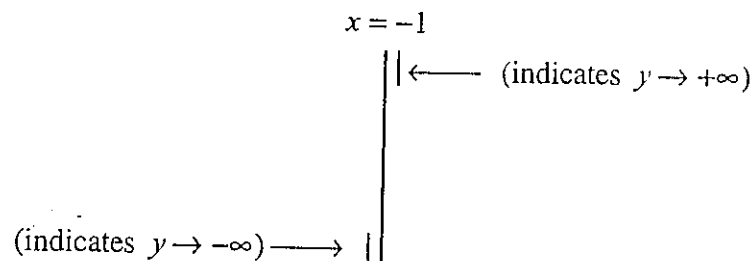
We can show this in an asymptote diagram.

$x = -4$

$\leftarrow$ (indicates $y \to +\infty$)

(indicates $y \to -\infty$) $\longrightarrow$

As $x \to -1^-$ (means x tends to -1 from a negative direction ie. $x < -1$)

$y \to \dfrac{(+)}{(+)(-)} \to -\infty$ (assign a sign to each part of the rational function and

find the <u>net</u> <u>sign</u>. $-\infty$ means negative infinity)

As $x \to -1^+$ (means x tends to -1 from a positive direction ie. $x > -1$)

$y \to \dfrac{(-)}{(+)(-)} \to +\infty$ (assign a sign to each part of the rational function and

find the <u>net</u> <u>sign</u>. $+\infty$ means positive infinity)

We can show this in an asymptote diagram.

$x = -1$

$\leftarrow$ (indicates $y \to +\infty$)

(indicates $y \to -\infty$) $\longrightarrow$

2.   (If the degree of the numerator = the degree of the denominator) :-

$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 5x + 4} = \frac{(x+1)^2}{(x+4)(x+1)}$$

<u>Vertical asymptotes occur at the zeros of the denominator</u>

$(x+4)(x+1) = 0$

$x = -4$ and $x = -1$ are asymptotes.

We now have to determine how the curve approaches these asymptotes.

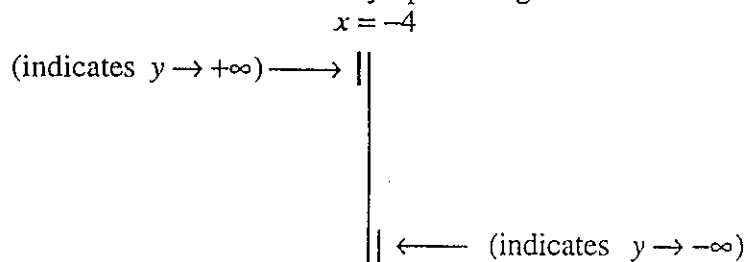As $x \to -4^-$ (means x tends to -4 from a negative direction ie. $x < -4$)

$y \to \dfrac{(+)}{(-)(-)} \to +\infty$ (assign a sign to each part of the rational function and

find the <u>net</u> <u>sign</u>. $+\infty$ means positive infinity)

As $x \to -4^+$ (means x tends to -4 from a positive direction ie. $x > -4$)

$y \to \dfrac{(+)}{(+)(-)} \to -\infty$ (put a sign to each part of the rational function and

                 find the <u>net</u> <u>sign</u>. $-\infty$ means negative infinity)
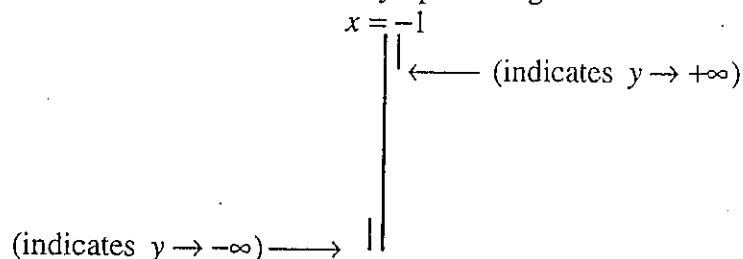
We can show this in an asymptote diagram.

$$x = -4$$

(indicates $y \to +\infty$) $\longrightarrow$ $\|$

                                   $\|$ $\longleftarrow$ (indicates $y \to -\infty$)

As $x \to -1^-$ (means x tends to -1 from a negative direction ie. $x < -1$)

$y \to \dfrac{(+)}{(+)(-)} \to -\infty$ (put a sign to each part of the rational function and

                 find the <u>net</u> <u>sign</u>. $-\infty$ means negative infinity)

As $x \to -1^+$ (means x tends to -1 from a positive direction ie. $x > -1$)

$y \to \dfrac{(+)}{(+)(+)} \to +\infty$ (put a sign to each part of the rational function and

                 find the <u>net</u> <u>sign</u>. $+\infty$ means positive infinity)

We can show this in an asymptote diagram.

$$x = -1$$

                         $\|$ $\longleftarrow$ (indicates $y \to +\infty$)

(indicates $y \to -\infty$) $\longrightarrow$ $\|$

3.   (If the degree of the numerator > the degree of the denominator) :-

$$f(x) = \frac{x^2 + 4x + 3}{x + 2} = \frac{(x+1)(x+3)}{x+2}$$

<u>Vertical asymptotes occur at the zeros of the denominator</u>

$$x + 2 = 0$$
$$x = -2 \text{ is an asymptote.}$$

We now have to determine how the curve approaches this asymptotes.

As $x \to -2^-$ (means x tends to -2 from a negative direction ie. $x < -2$)

$y \to \dfrac{(-)(+)}{(-)} \to +\infty$ (put a sign to each part of the rational function and

                 find the <u>net</u> <u>sign</u>. $+\infty$ means negative infinity)

                                              cont'd ....

As $x \to -2^+$ (means x tends to -2 from a positive direction ie. $x > -2$)

$y \to \dfrac{(-)(+)}{(+)} \to -\infty$ (put a sign to each part of the rational function and

find the <u>net sign</u>. $-\infty$ means positive infinity)

We can show this in an asymptote diagram.

$x = -2$

(indicates $y \to +\infty$) $\longrightarrow$

$\longleftarrow$ (indicates $y \to -\infty$)
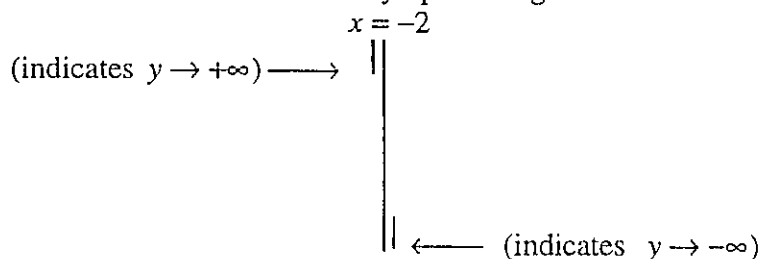
Exercise 1

Find all the vertical asymptotes of the following rational functions.

1. $f(x) = \dfrac{4}{x-2}$

2. $f(x) = \dfrac{3x-1}{x^2 + 2x - 3}$

3. $f(x) = \dfrac{12}{x^2 - 2x - 3}$

4. $f(x) = \dfrac{x+4}{x-2}$

5. $f(x) = \dfrac{x^2}{4 - x^2}$

6. $f(x) = \dfrac{x(x+1)}{(x-1)(x+2)}$

7. $f(x) = \dfrac{(x-1)(x-4)}{x}$

8. $f(x) = \dfrac{x^2 + 3}{x-1}$

9. $f(x) = \dfrac{x^3}{x^2 + 3}$

10. $f(x) = \dfrac{x}{x^2 + 4}$

11. $f(x) = \dfrac{x^2}{x-1}$

12. $f(x) = \dfrac{2x^2}{x^2 - 1}$

Find the non-vertical asymptote of a rational function.

The **non - vertical asymptote** is in the form $y = c$ or $y = mx + c$.
The way the curve approaches the asymptotes must also be determined.

Examples

1. (Note the degree of the numerator $<$ the degree of the denominator)

$$f(x) = \dfrac{2x+3}{x^2 + 5x + 4} = \dfrac{2x+3}{(x+4)(x+1)}$$

<u>Divide each term of the numerator and denominator by the highest power of x.</u>

ie. divide by x^2 $\qquad f(x) = \dfrac{\dfrac{2}{x} + \dfrac{3}{x^2}}{1 + \dfrac{5}{x} + \dfrac{4}{x^2}}$

As $x \to \pm\infty$ (ie large positive and negative values of x, $\dfrac{2}{x}$, $\dfrac{3}{x^2}$, $\dfrac{5}{x}$ and $\dfrac{4}{x^2}$,

all tend to **zero**.)

As a result, $f(x) \to 0$ and so $y = 0$ is a horizontal asymptote. (the x - axis !)
Now we must find out if the curve approaches this asymptote from <u>above or below</u>.

cont'd ....

Proceed as follows.

As $x \to +\infty$, $y \to 0^+$ (above the value of 0, since the fraction will be very small and positive)

As $x \to -\infty$, $y \to 0^-$ (below the value of 0, since the fraction $\dfrac{2}{x}$ is negative and

larger than the fraction $\dfrac{3}{x^2}$, making the numerator negative.

At the same time, the denominator becomes
1 – fraction + fraction making the <u>net sign</u> positive.)

We can show this in an asymptote diagram.

Indicates $y \to 0^-$ Indicates $y \to 0^+$
(from below)   (from above)

2. (If the degree of the numerator = the degree of the denominator) :-

$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 5x + 4} = \frac{(x+1)^2}{(x+4)(x+1)}$$

<u>Divide the numerator by the denominator</u> :-

$$\begin{array}{r} 1 \\ x^2 + 5x + 4 \,\big)\, x^2 + 2x + 1 \\ \underline{x^2 + 5x + 4} \\ -3x - 3 \end{array}$$

Write as $f(x) = 1 - \dfrac{(3x+3)}{x^2 + 5x + 4}$ using long division

Now divide each term of $\dfrac{(3x+3)}{x^2 + 5x + 4}$ by the highest power of x

ie. divide by x^2

$$f(x) = 1 - \frac{\left(\dfrac{3}{x} + \dfrac{3}{x^2}\right)}{\left(1 + \dfrac{5}{x} + \dfrac{4}{x^2}\right)}$$

As $x \to \pm\infty$ (ie large positive and negative values of x, $\dfrac{3}{x}$, $\dfrac{3}{x^2}$, $\dfrac{5}{x}$ and $\dfrac{4}{x^2}$, all tend to **zero**.)

As a result, $f(x) \to 1$ and so $y = 1$ is a horizontal asymptote.

Now we must find out if the curve approaches this asymptote from above or below.

Proceed as follows.

As $x \to +\infty$, $y \to 1^-$ (below the value of 1, since the fraction will be very small and positive)

As $x \to -\infty$, $y \to 1^+$ (above the value of 1, since the fraction $\dfrac{3}{x}$ is negative and

<u>larger</u> than the fraction $\dfrac{3}{x^2}$, making the numerator negative.

At the same time, the denominator becomes approximately 1.
$f(x) = 1 - (^-/_1) \Rightarrow 1^+ \Rightarrow$ slightly bigger than 1.

We can show this in an asymptote diagram.

Indicates $y \to 1^+$ Indicates $y \to 1^-$
(from above)   (from below)

3. (If the degree of the numerator > the degree of the denominator) :-

$$f(x) = \frac{x^2 + 4x + 3}{x + 2} = \frac{(x+1)(x+3)}{x+2}$$

Divide the numerator by the denominator :-

Write as $f(x) = x + 2 - \dfrac{1}{x+2}$ using long division

$$\begin{array}{r} x + 2 \\ x+2 \overline{\smash{\big)} x^2 + 4x + 3} \\ \underline{x^2 + 2x} \\ 2x + 3 \\ \underline{2x + 4} \\ -1 \end{array}$$

Now divide each term of $\dfrac{1}{x+2}$ by the highest power of x.

ie. divide by x $f(x) = x + 2 - \dfrac{\frac{1}{x}}{1 + \frac{2}{x}}$

As $x \to \pm\infty$ (ie large positive and negative values of x, $\dfrac{1}{x}$ and $\dfrac{2}{x}$ tend to **zero**.)

As a result, $f(x) \to x + 2$ and so $f(x) = x + 2$ is a oblique asymptote. (slanting)
Now we must find out if the curve approaches this asymptote from above or below.
Proceed as follows.

As $x \to +\infty$, $y \to (x+2)^-$ (below the line $f(x) = x + 2$, since the fraction will be very small and positive)

As $x \to -\infty$, $y \to (x+2)^+$ (above the line $f(x) = x + 2$, since the fraction $\dfrac{1}{x}$ is negative and the denominator becomes $1 -$ fraction making the net sign positive.)

We can show this in an asymptote diagram.

(Indicates $y \to (x+2)^-$)

(Asymptote)

(Indicates $y \to (x+2)^+$)
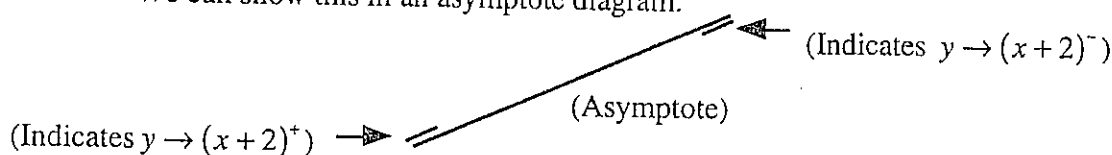
<u>Exercise 2</u>

Find the non - vertical asymptote of the following rational functions.

1. $f(x) = \dfrac{4}{x - 2}$

2. $f(x) = \dfrac{3x - 1}{x^2 + 2x - 3}$

3. $f(x) = \dfrac{12}{x^2 - 2x - 3}$

4. $f(x) = \dfrac{x + 4}{x - 2}$

5. $f(x) = \dfrac{x^2}{4 - x^2}$

6. $f(x) = \dfrac{x(x+1)}{(x-1)(x+2)}$

7. $f(x) = \dfrac{(x-1)(x-4)}{x - 2}$

8. $f(x) = \dfrac{x^2 + 3}{x - 1}$

9. $f(x) = \dfrac{x^3}{x^2 + 3}$

10. $f(x) = \dfrac{x}{x^2 + 4}$

11. $f(x) = \dfrac{x^2}{x - 1}$

12. $f(x) = \dfrac{2x^2}{x^2 - 1}$

Sketch the graph of a rational function including appropriate analysis of stationary points.

Curve Sketching

1   **Sketching the graph of a rational function using calculus**

(a)   Find all the stationary points and their nature.

(b)   Find all the asymptotes and investigate the approach of the curve to each.

(c)   Find all the crossings of the y axis. and the x axis (if easily found)

Note   The table of signs may be easier than finding the second derivative in order to determine the nature of the stationary points.

Examples   Sketch the following curves.

1. $f(x) = \dfrac{2x^2 + x - 1}{x - 1} = \dfrac{(x+1)(2x-1)}{x-1}$

(a)   **Stationary points and their nature.**

$$f'(x) = \frac{(x-1)(4x+1) - (2x^2 + x - 1) \times 1}{(x-1)^2} = \frac{4x^2 - 3x - 1 - 2x^2 - x + 1}{(x-1)^2} = \frac{2x^2 - 4x}{(x-1)^2}$$

For S.V. $f'(x) = 0$

$2x^2 - 4x = 0$

$2x(x - 2) = 0$

$x = 0$ and $x = 2$

$y = 1$ $y = 9$

Stationary points at $(0,1)$ and $(2,9)$

$$f''(x) = \frac{(x-1)^2 \times (4x-4) - (2x^2 - 4x) \times 2(x-1)}{(x-1)^4} = \frac{4(x-1)^3 - 4x(x-1)(x-2)}{(x-1)^4}$$

$$= \frac{4(x-1)^2 - 4x(x-2)}{(x-1)^3}$$

$f''(0) = \dfrac{(+) - 0}{(-)} = (-)$ ie. $(0, 1)$ is a Maximum Turning point.

$f''(2) = \dfrac{(+) - 0}{(+)} = (+)$ ie. $(2, 9)$ is a Minimum Turning point.

(b)   **Asymptotes.**

(i)   **Vertical.**

Vertical asymptotes occur at the zeros of the denominator

$x - 1 = 0$.

$x = 1$ is an asymptote. $x = 1$

As $x \to 1^-$, $y \to \dfrac{(+)(+)}{(-)} \to -\infty$

As $x \to 1^+$, $y \to \dfrac{(+)(+)}{(+)} \to +\infty$

(ii)   **Non-vertical.**
Divide the numerator by the denominator :-

Write as $f(x) = 2x + 3 + \dfrac{2}{x-1}$ using long division

$$\begin{array}{r} 2x+3 \\ x-1 \overline{\smash{\big)}\, 2x^2 + x - 1} \\ \underline{2x^2 - 2x} \\ 3x - 1 \\ \underline{3x - 3} \\ +2 \end{array}$$

$$f(x) = 2x + 3 + \frac{\frac{2}{x}}{1 - \frac{1}{x}}$$

As $x \to \pm\infty$, $f(x) \to 2x + 3$ and so $f(x) = 2x + 3$ is a slant asymptote.
As $x \to +\infty$, $y \to (2x+3)^+$ (above)
As $x \to -\infty$, $y \to (2x+3)^-$ (below)

$y = 2x + 3$

(c)   **Axes crossings.**

When $f(x) = 0$, $(x+1)(2x-1) = 0$ ie. $x = -1$ and $x = \dfrac{1}{2}$.

The graph crosses the x-axis at $(-1, 0)$ and $\left(\dfrac{1}{2}, 0\right)$.

When $x = 0$, $y = 1$.
The graph crosses the y-axis at $(0, 1)$.

Sketch.

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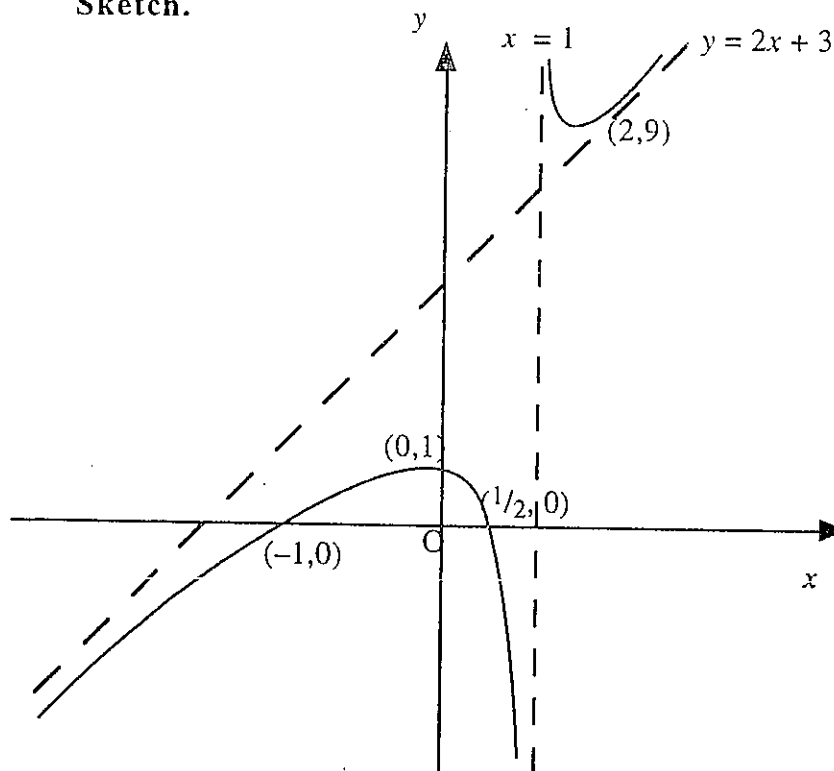
2. $f(x) = \dfrac{2x^2 + 4x + 3}{x^2 - 1} = \dfrac{2x^2 + 4x + 3}{(x+1)(x-1)}$

(a)   **Stationary points and their nature.**

$$f'(x) = \frac{(x^2-1)(4x+4) - (2x^2+4x+3)\times 2x}{(x^2-1)^2} = \frac{4x^3 + 4x^2 - 4x - 4 - 4x^3 - 8x^2 - 6x}{(x^2-1)^2}$$

$$= \frac{-4x^2 - 10x - 4}{(x^2-1)^2}$$

For S.V. $f'(x) = 0$

$-4x^2 - 10x - 4 = 0$

$-2(x+2)(2x+1) = 0$

$x = -2$ and $x = -\dfrac{1}{2}$

$y = 1$ $y = -2$

Stationary points at $(-2, 1)$ and $(-\dfrac{1}{2}, -2)$

$$f''(x) = \frac{(x^2-1)^2(-8x-10) - (-4x^2-10x-4)4x(x^2-1)}{(x^2-1)^4}$$

$$= \frac{-2(x^2-1)(4x+5) + 8x(x+2)(2x+1)}{(x^2-1)^3}$$

$f''(-2) = \dfrac{(+)+0}{(+)} = (+)$ ie. $(-2, 1)$ is a Minimum Turning point.

$f''\left(-\dfrac{1}{2}\right) = \dfrac{(+)+0}{(-)} = (-)$ ie. $\left(-\dfrac{1}{2}, -2\right)$ is a Maximum Turning point.

(b)   **Asymptotes.**

(i)   **Vertical.**
      Vertical asymptotes occur at the zeros of the denominator
      $(x+1)(x-1) = 0$
      $x = -1$ and $x = 1$ are asymptotes.

      As $x \to -1^-$, $y \to \dfrac{(+)}{(-)(-)} \to +\infty$

      As $x \to -1^+$, $y \to \dfrac{(+)}{(+)(-)} \to -\infty$

      $x = -1$

      As $x \to 1^-$, $y \to \dfrac{(+)}{(+)(-)} \to -\infty$

      As $x \to 1^+$, $y \to \dfrac{(+)}{(+)(+)} \to +\infty$

      $x = 1$

(ii)   **Non-vertical.**

Divide the numerator by the denominator :-

Write as $f(x) = 2 + \dfrac{4x+5}{x^2-1}$ using long division

$$x^2-1 \overline{)\begin{array}{r} 2 \\ 2x^2+4x+3 \\ 2x^2 -2 \\ \hline 4x+5 \end{array}}$$

$$f(x) = 2 + \dfrac{\dfrac{4}{x}+\dfrac{5}{x^2}}{1-\dfrac{1}{x^2}}$$

As $x \to \pm\infty$, $f(x) \to 2$ and so $f(x) = 2$ is a horizontal asymptote.

As $x \to +\infty$, $y \to 2^+$ (above)

As $x \to -\infty$, $y \to 2^-$ (below) $y = 2$

(c)   **Axes crossings.**

When $f(x) = 0$, $2x^2+4x+3 = 0$ ie has no real roots since $b^2-4ac < 0$.
The graph does not cross the x-axis .
When $x = 0$, $y = -3$.
The graph crosses the y-axis at $(0,-3)$.

Sketch.
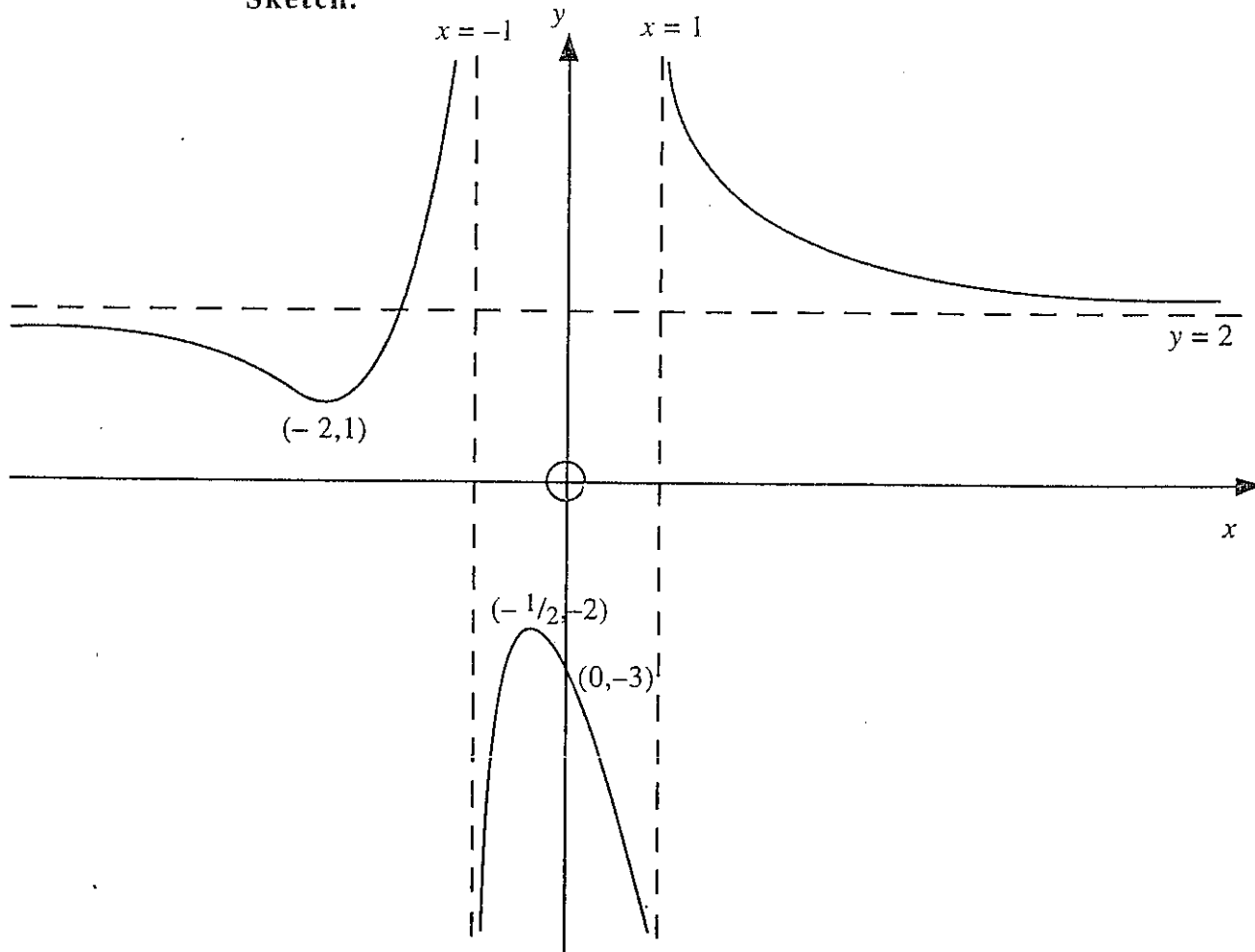
Exercise 3

Sketch the following curves.

1. $f(x) = \dfrac{4}{x-2}$ 2. $f(x) = \dfrac{x-2}{x-1}$

3. $f(x) = \dfrac{x^2+x-2}{x^2+x-6}$ 4. $f(x) = \dfrac{x^2+2x+5}{x+1}$

5. $f(x) = \dfrac{x+1}{x^2+2x+5}$ 6. $f(x) = \dfrac{2x^2}{x^2-1}$

7. $f(x) = \dfrac{x^2-10x+9}{x^2+10x+9}$ 8. $f(x) = \dfrac{2x^2-3x-3}{x^2-3x+2}$

2   Sketching the graph of a rational function without using calculus

(a)   Find the crossings of the axes.

(b)   Find the equations of the asymptotes and investigate how the curve approaches each.

(c)   Find the **range** of values of y.

Examples

1.   Sketch the curve $y = \dfrac{3x+3}{x(3-x)}$

(a)   At $y = 0$, $\dfrac{3x+3}{x(3-x)} = 0$ $\Rightarrow 3x+3 = 0$ $\Rightarrow x = -1$ ie. $(-1, 0)$

At $x = 0$, $y = \dfrac{3}{0}$ (undefined)   ie. does not cross the y - axis

(b)   **Vertical asymptotes** at $x(3-x) = 0$ ie. $x = 0$, $x = 3$

$x = 0$

As $x \to 0^-, y \to \dfrac{(+)}{(-)(+)} \to -\infty$

As $x \to 0^+, y \to \dfrac{(+)}{(+)(+)} \to +\infty$

As $x \to 3^-, y \to \dfrac{(+)}{(+)(+)} \to +\infty$

As $x \to 3^+, y \to \dfrac{(+)}{(+)(-)} \to -\infty$

$x = 3$

Horizontal/sloping asymptotes

$$y = \frac{3x+3}{3x-x^2} = \frac{\dfrac{3x}{x^2}+\dfrac{3}{x^2}}{\dfrac{3x}{x^2}-\dfrac{x^2}{x^2}} = \frac{\dfrac{3}{x}+\dfrac{3}{x^2}}{\dfrac{3}{x}-1}$$

As $x \to \pm\infty$, $y \to 0$ and so $y = 0$ is a horizontal asymptote.

As $x \to +\infty, y \to \dfrac{(+)}{-1} \to 0^-$ $y = 0$

As $x \to -\infty, y \to \dfrac{(-)}{-1} \to 0^+$

(c) **Range of the values of y**

From $y = \dfrac{3x + 3 = 0}{x(3 - x)}$

$3x + 3 = y(3x - x^2) \implies 3x + 3 = 3xy - x^2 y$

$x^2 y + 3x - 3xy + 3 = 0 \implies x^2 y + 3x(1 - y) + 3 = 0$

$x^2 y + 3x(1 - y) + 3 = 0$ is a quadratic equation in x.

Thus for real values of x, $b^2 - 4ac \geq 0$.

ie. $[3(1 - y)]^2 - 12y \geq 0$ or $9y^2 - 30y + 9 \geq 0$ or $3(3y - 1)(y - 3) \geq 0$

ie. $y \leq \dfrac{1}{3}$ and $y \geq 3$ for real x.

This means that no part of the graph can lie between $y = \dfrac{1}{3}$ and $y = 3$.

The curve so far looks like this.
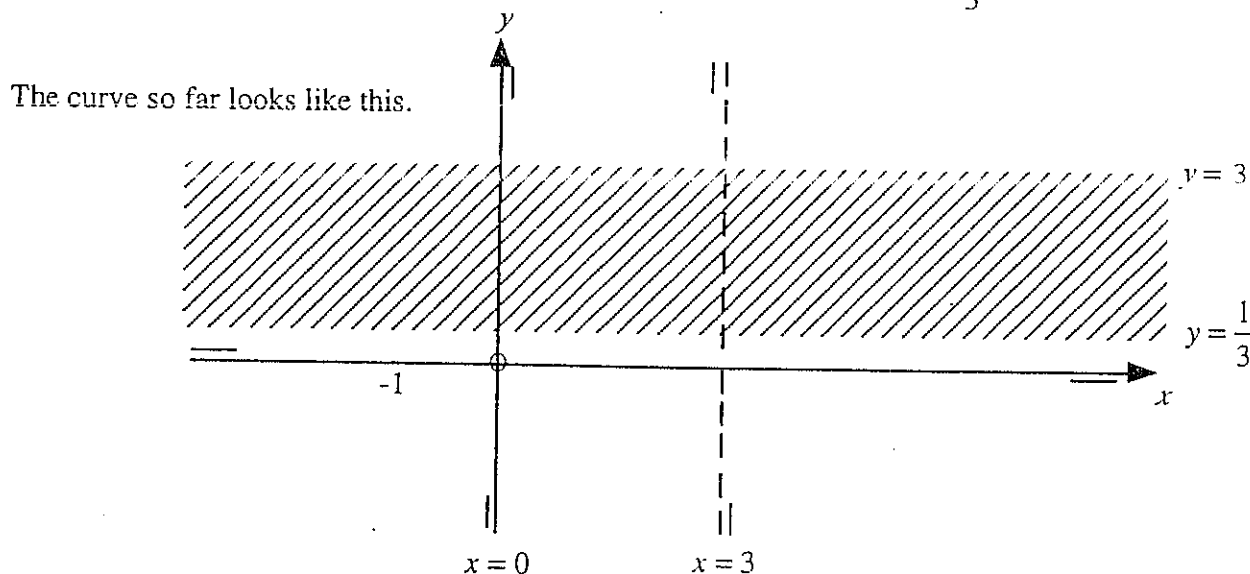
We expect a (local) maximum for $y \leq \dfrac{1}{3}$ and a (local) minimum for $y \geq 3$.

The coordinates of the turning points can be found by substitution of $y = \dfrac{1}{3}$ and $y = 3$ into $x^2 y + 3x(1 - y) + 3 = 0$ to find the corresponding values of x ie. -3 and 1.

The complete graph is as follows
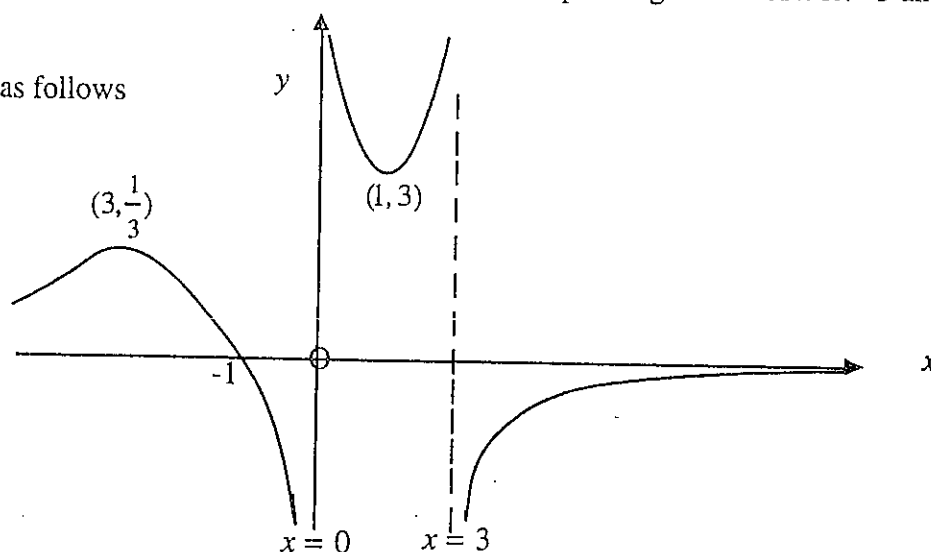
2.   Sketch $y = \dfrac{12}{x^2 + 2x - 3} = \dfrac{12}{(x+3)(x-1)}$

(a)   When $y = 0$, $\dfrac{12}{(x+3)(x-1)} = 0$ (No solution ie. does not cross the x-axis)

When $x = 0$, $y = -4$ Cosses the y-axis at $(0, -4)$

(b)   **Vertical asymptotes** at $(x+3)(x-1) = 0$ ie. $x = -3$ and $x = 1$

$x = -3$

As $x \rightarrow -3^-, y \rightarrow \dfrac{(+)}{(-)(-)} \rightarrow + \infty$

As $x \rightarrow -3^+, y \rightarrow \dfrac{(+)}{(+)(-)} \rightarrow - \infty$

$x = 1$

As $x \rightarrow 1^-, y \rightarrow \dfrac{(+)}{(+)(-)} \rightarrow - \infty$

As $x \rightarrow 1^+, y \rightarrow \dfrac{(+)}{(+)(+)} \rightarrow + \infty$

Horizontal/slant asymptotes

$$y = \dfrac{12}{x^2 + 2x - 3} = \dfrac{\frac{12}{x^2}}{\frac{x^2}{x^2} + \frac{2x}{x^2} - \frac{3}{x^2}} = \dfrac{\frac{12}{x^2}}{1 + \frac{2x}{x^2} - \frac{3}{x^2}}$$

As $x \rightarrow \pm\infty$, $y \rightarrow 0$ and so $y = 0$ is a horizontal asymptote.

As $x \rightarrow +\infty, y \rightarrow \dfrac{(+)}{1} \rightarrow 0^+$

As $x \rightarrow -\infty, y \rightarrow \dfrac{(+)}{1} \rightarrow 0^+$

$y = 0$

(c)   **Range of the values of y**

From $y = \dfrac{12}{x^2 + 2x - 3}$

$y(x^2 + 2x - 3) = 12 \Longrightarrow x^2 y + 2xy - 3y - 12$

$x^2 y + 2xy - (3y + 12) = 0$

$x^2 y + 2xy - (3y + 12) = 0$ is a quadratic equation in x.

Thus for real values of x , $b^2 - 4ac \geq 0$.

ie. $4y^2 + 4y(3y + 12) \geq 0$

$16y^2 + 48y \geq 0$

$16y(y + 3) \geq 0$

Solve by either a table of signs or sketching the quadratic graph.

ie. $y \leq -3$ and $y \geq 0$

This means that no part of the graph can lie between $y = -3$ and $y = 0$
The curve so far looks like this.
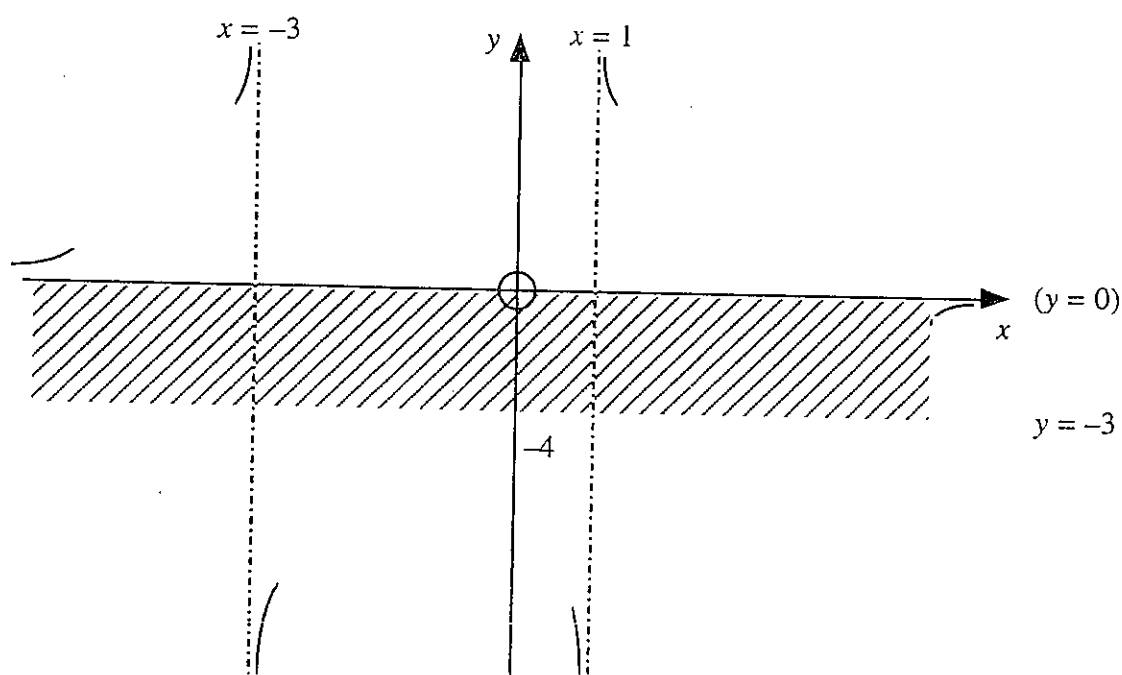
For $y \le -3$, there is a (local) maximum.

The coordinates of the turning point can be found by substituting $y = -3$ into

$x^2y + 2xy - (3y + 12) = 0$ to find the corresponding value of x i.e. $x = -1$

The completed curve looks as follows.
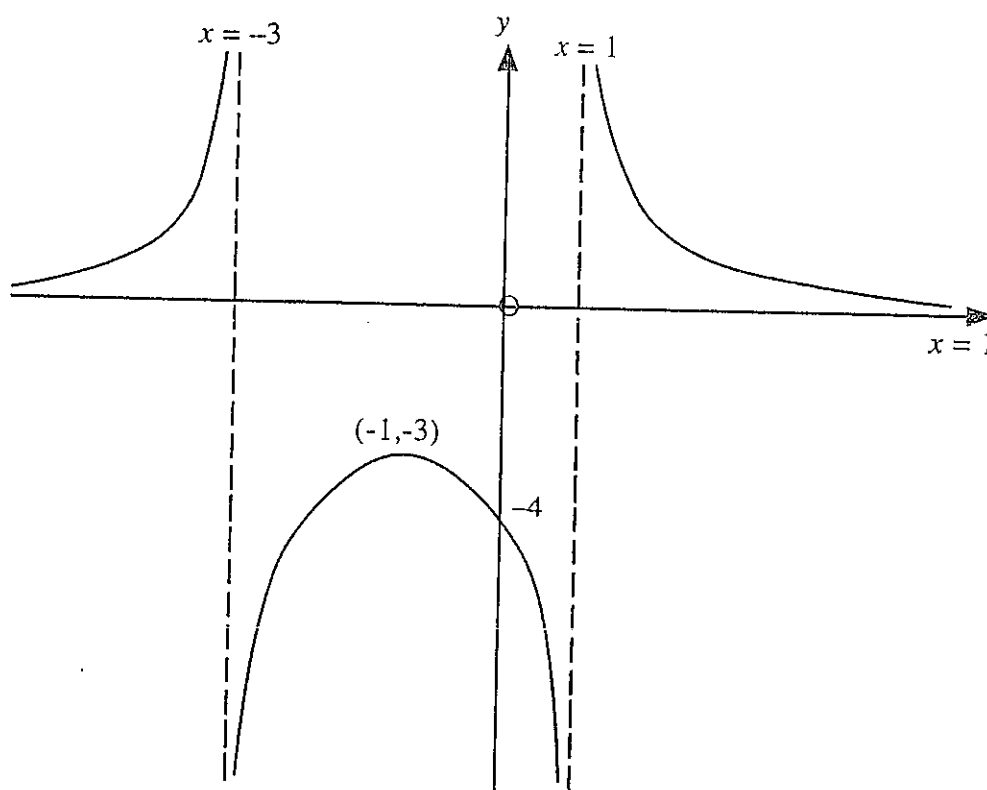
Exercise 4

Sketch the following curves.

1. $y = \dfrac{4x - 5}{x^2 - 1}$

2. $y = \dfrac{x^2 + 3}{x - 1}$

3. $y = \dfrac{4(x + 1)}{x^2 + 2x + 2}$

4. $y = \dfrac{2x - 1}{(x - 1)^2}$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

Page 93 Exercise 2.3:1 Questions 1, 2,7,18,19 and 22

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 342 Exercise 14A Questions 9-19.

Given the graph of a function, sketch the graphs of a related function.

(a) **The graph of inverse functions**

The graph of the inverse of simple functions has already been met at Higher level.

e.g. (i) $f(x) = 2x - 3$ has an inverse of $f^{-1}(x) = \dfrac{1}{2}(x + 3)$ or $\dfrac{1}{2}x + \dfrac{3}{2}$

The graphs are as follows -:

$y = \frac{1}{2}x + 3$

$y = x$

$y = 2x - 3$
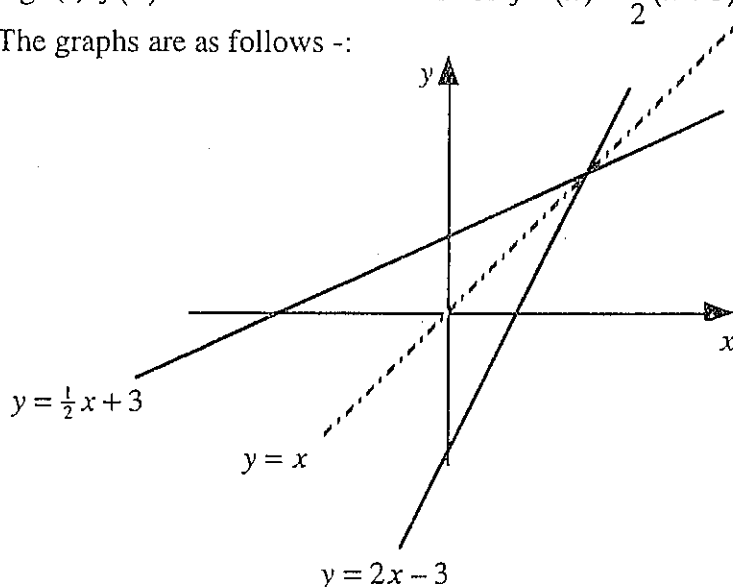
It has already been shown that the graph of an inverse function can be found by reflecting the graph of the function in the line $y = x$.

(ii) $f(x) = e^x$ has an inverse of $f^{-1}(x) = \ln x$

The graphs are as follows -:
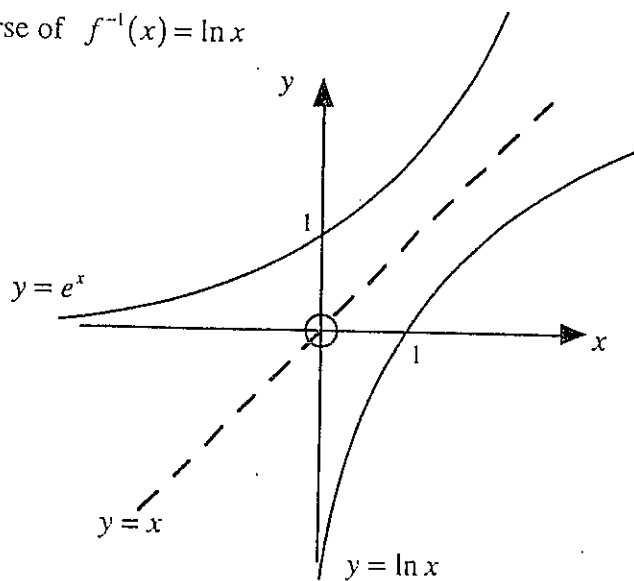
(iii) $f(x) = \sin x$ has an inverse of $f^{-1}(x) = \sin^{-1} x$

The graphs are as follows -:
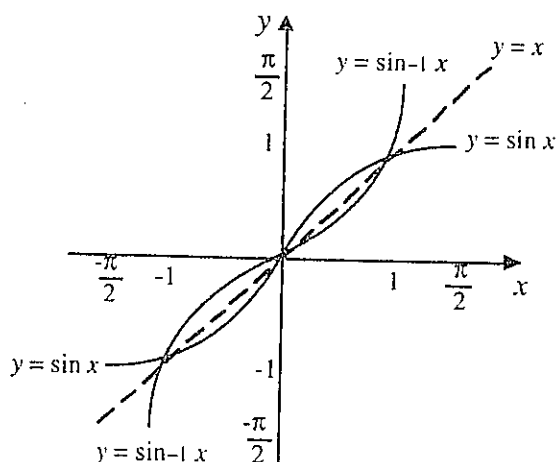
The inverse Sine Function is denoted by $\sin^{-1} x$ and read as sine inverse x or arc sine x or sine to the minus 1 of x.

The inverse sine function, $\sin^{-1} x$, is defined as the **angle** whose sine is x.

If we consider the graph of $y = \sin x$, we see that there is an infinite number of angles whose sine could be x.

Consequently, in order that the inverse sine should be a true function, we must restrict the angle concerned.

We choose the simplest possible restriction for the angle - the closed interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$

A revised definition is therefore that the inverse sine function, $\sin^{-1} x$, is the angle, in the closed interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$, whose sine is x.

Examples

Evaluate :-

1. $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$

2. $\sin^{-1}(-1) = -\dfrac{\pi}{2}$

3. $\sin^{-1}\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$

4. $\sin^{-1}(0) = 0$

(iv) $f(x) = \cos x$ has an inverse of $f^{-1}(x) = \cos^{-1} x$

The graphs are as follows -:
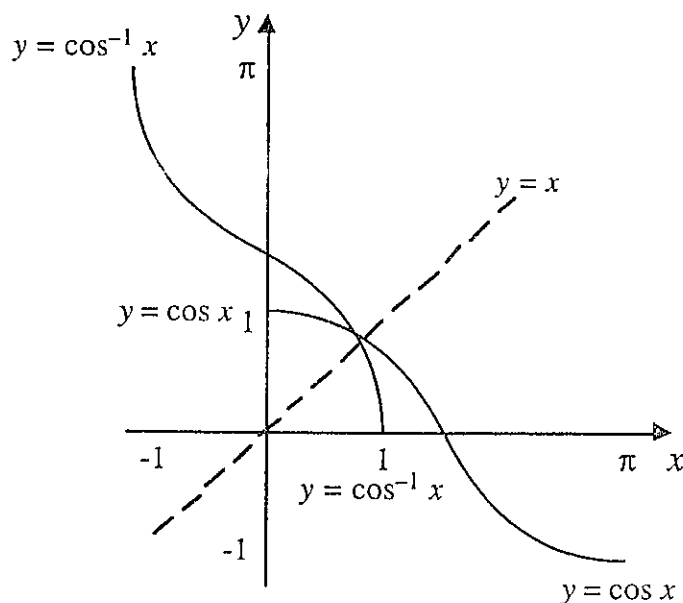
The inverse Cosine Function is denoted by $\cos^{-1} x$ and is read as cos inverse x or arc cos x or cos to the minus 1 of x.

In this case , we restrict the angle to the closed interval $[0, \pi]$.

The inverse cosine function is defined as the angle , in the closed interval $[0, \pi]$, whose cosine is x.

Examples

Evaluate :-

1. $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{6}$

2. $\cos^{-1}(-1) = \pi$

(v) $f(x) = \tan x$ has an inverse of $f^{-1}(x) = \tan^{-1} x$.

The graphs are as follows -:
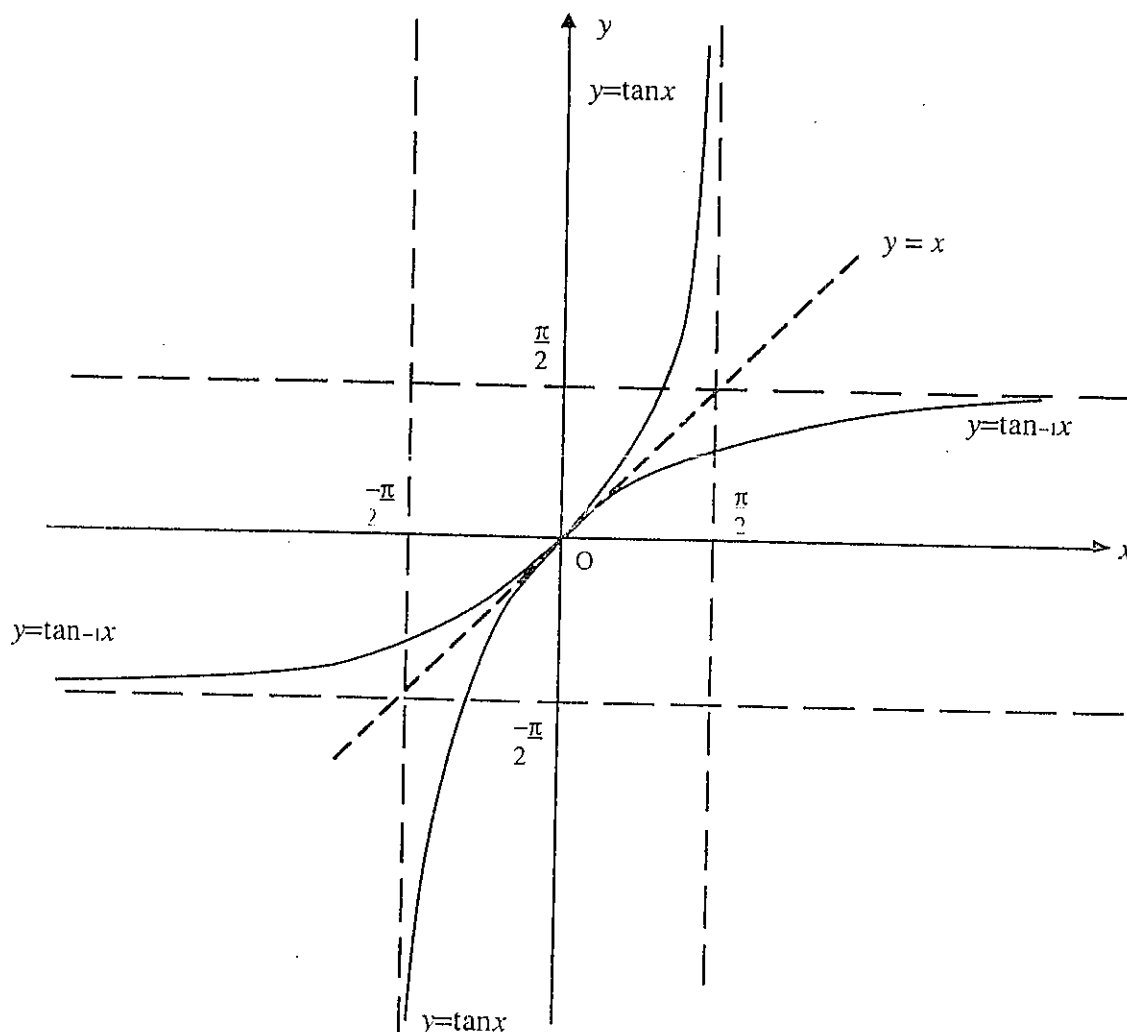
The inverse Tangent function is denoted by $\tan^{-1} x$ and is read as tan inverse x or arc tan x. In this case we restrict the angle to the open interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

The inverse tangent function is defined as the angle in the open interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ whose tangent is x.

Examples

Evaluate :-

1. $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{3}$ 2. $\tan^{-1}(-1) = -\dfrac{\pi}{4}$

The following relationships on the next page were met at Higher level

cont'd ....

(b)   <u>The graph of $kf(x)$</u>
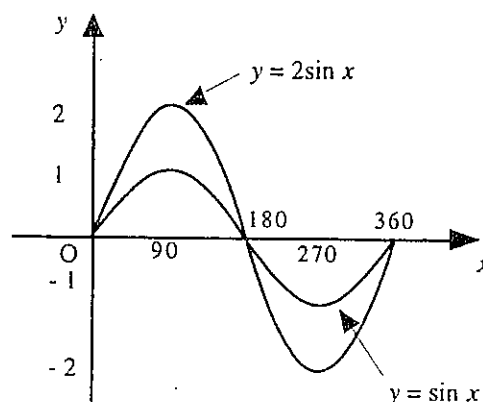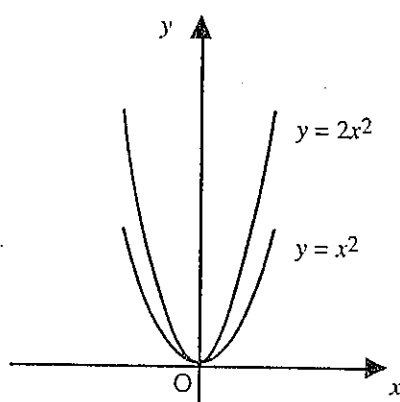
The graph of $f(x)$ is stretched vertically if $k > 1$ and compressed vertically if $0 < k < 1$.

(c)   <u>The graph of $f(x) + k$</u>
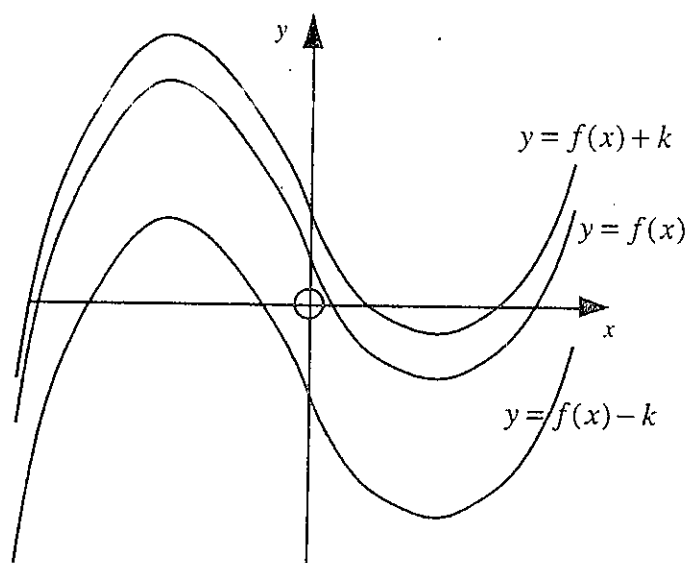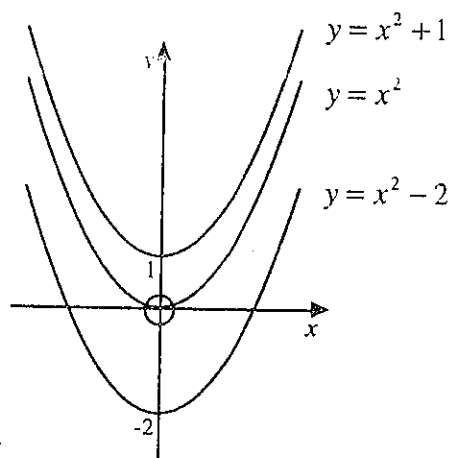
The graph of $f(x)$ is moved vertically upwards if $k > 0$ and downwards if $k < 0$.

(d)   <u>The graph of $f(x + k)$</u>
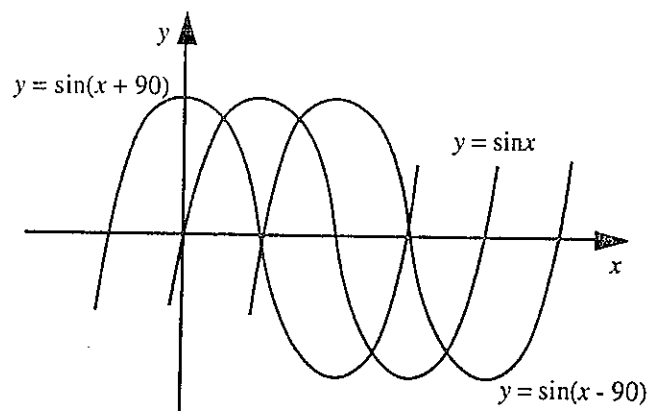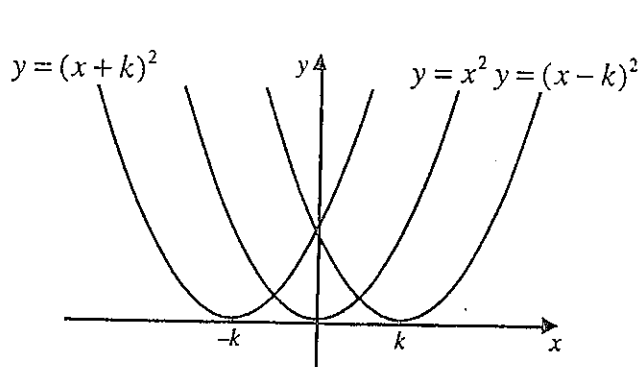
The graph of $f(x)$ is moved horizontally left if $k > 0$ and right if $k < 0$.

(e)   **The graph of f(kx)**

$y = x^2 - 4$ $y = (2x)^2 - 4$

$y = \sin x$ $y = \sin 2x$
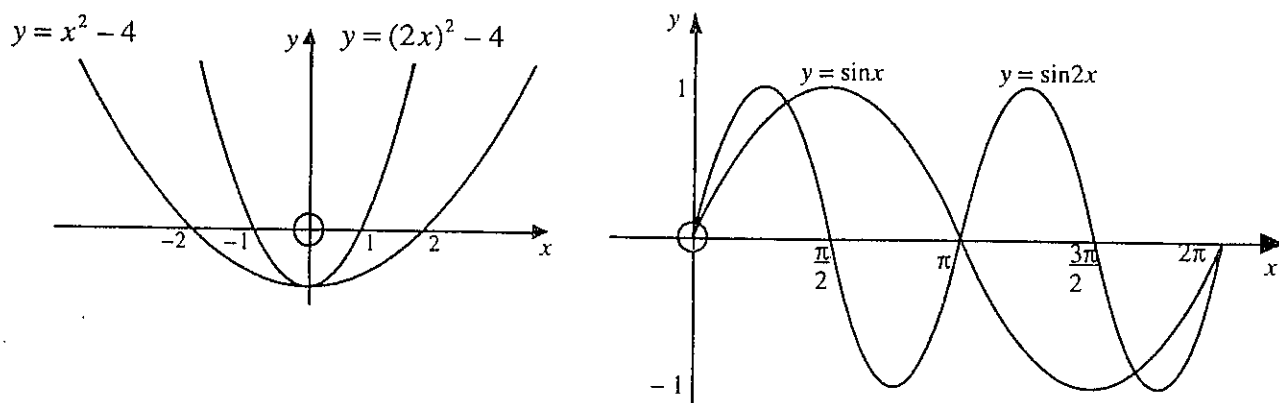
The graph of f(x) is compressed if k > 1 and stretched if k < 1.

(f)   **The graph of If(x)I   - The Modulus Function**

(i)   When f(x) = x , f(x) is negative when x is negative.

But if f(x) = IxI , f takes the **positive** numerical value of x.

e.g. when x = – 3 , IxI = 3 so f(x) is always positive.

The graph of f(x) = IxI can be obtained from the graph of f(x) = x by simply flipping any parts of the graph of f(x), which appear below the x – axis, over the x – axis.

$f(x) = x$ $f(x) = IxI$
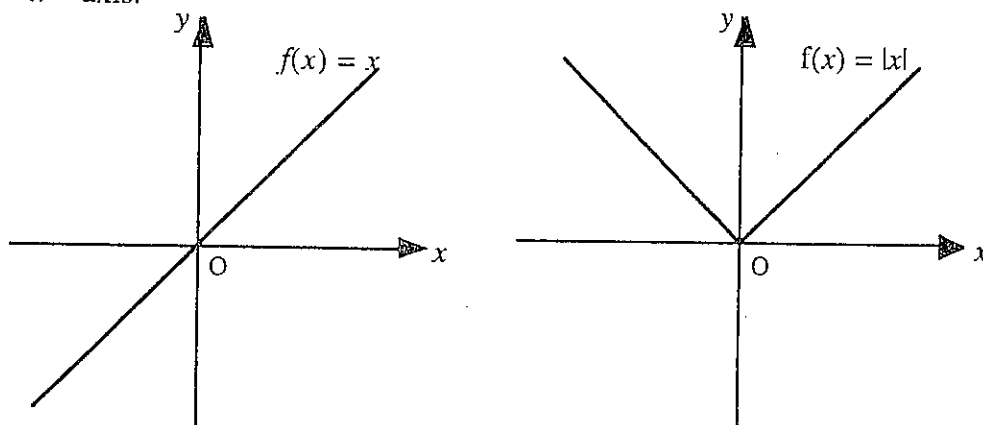
(ii)   To sketch f(x) = I(x – 1)(x – 2)I , we start by sketching f(x) = (x – 1)(x – 2). We then reflect in the x - axis the p   of this curve which is below the x – axis.

f(x) =(x - 1)(x - 2) $f(x) = I(x - 1)(x - 2)I$
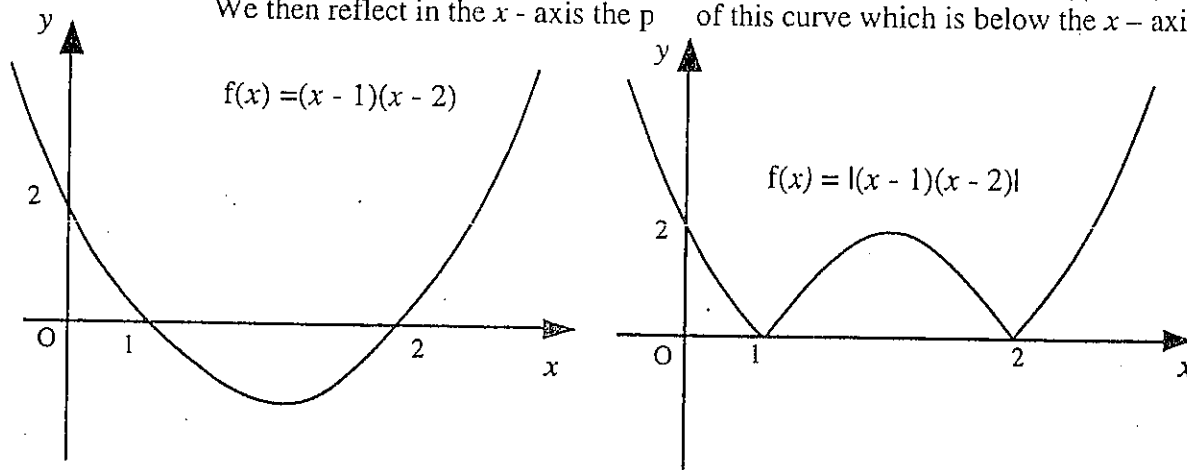
(g) Odd and Even functions

An **even** function is any function whose curve has the y - axis as a line of symmetry. Curves, having only even powers of x, are symmetrical about the y - axis.

e.g.
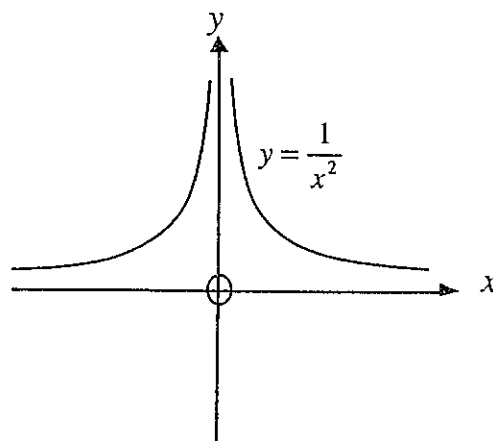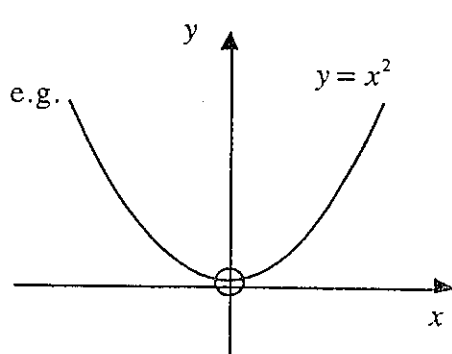
An alternative method of defining an **even** function is to show that $f(-x) = f(x)$ for **all** values of x.

e.g. (i) $f(x) = x^2$

$f(-x) = (-x)^2 = f(x)$

(ii) $f(x) = \dfrac{1}{x^2}$

$f(-x) = \dfrac{1}{(-x)^2} = \dfrac{1}{x^2} = f(x)$

An **odd** function is any function whose curve has $180°$ rotational symmetry about the origin. Curves having only odd powers of x have $180°$ rotational symmetrical about the origin.

e.g.
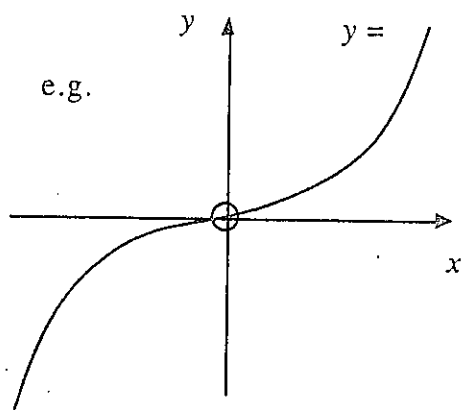
An alternative method of defining an **odd** function is to show that $f(-x) = -f(x)$. for **all** values of x.

e.g. $f(x) = x^3$
$f(-x) = (-x)^3 = -x^3 = -f(x)$

Exercise 5

1. Write down the equations of the inverses of the following functions :

   (a) $f(x) = 2x$
   (b) $f(x) = 2 - x$
   (c) $f(x) = {}^2/x$

   (d) $f(x) = 2^x$
   (e) $f(x) = 1 - 2x$
   (f) $f(x) = \ln(x - 2)$

2. Evaluate :

   (a) $\sin^{-1}(\sqrt{3}/2)$
   (b) $\tan^{-1}(1/\sqrt{3})$
   (c) $\tan^{-1}(1)$

   (d) $\sin^{-1}(1/2)$
   (e) $\cos^{-1}(-\sqrt{3}/2)$
   (f) $\tan^{-1}(\sqrt{3})$

3. Sketch the following graph of a function $y = f(x)$ for each part of the following question.

   On separate graphs, sketch the graphs of :

   (a) $f(x - 3)$
   (b) $f(x + 3)$

   (c) $f(x) + 2$
   (d) $2f(x)$

   (e) $-f(x)$
   (f) $f(-x)$
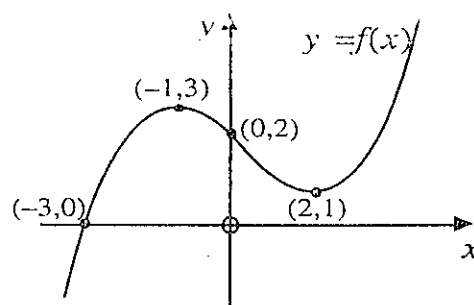
4. Sketch the graphs of $f(x)$ and $|f(x)|$.

   (a) $f(x) = x + 2$
   (b) $f(x) = 5 - 2x$
   (c) $f(x) = x^2 - 2x - 3$

   (d) $f(x) = 3x - x^2$
   (e) $f(x) = x^3 + 1$
   (f) $f(x) = \dfrac{1}{x} - 2$

5. Which of the following functions are odd, even or neither ?

   (a) $f(x) = (x + 4)(x - 2)$
   (b) $f(x) = 3x^2 + 5$

   (c) $f(x) = 2x - x^3$
   (d) $f(x) = \sin 2x$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
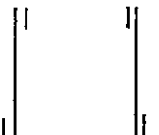
Various sections on graphs.Page 105, 108, 128, 163.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
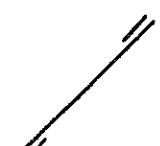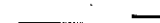
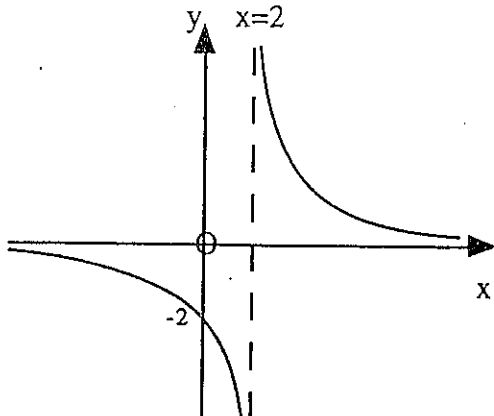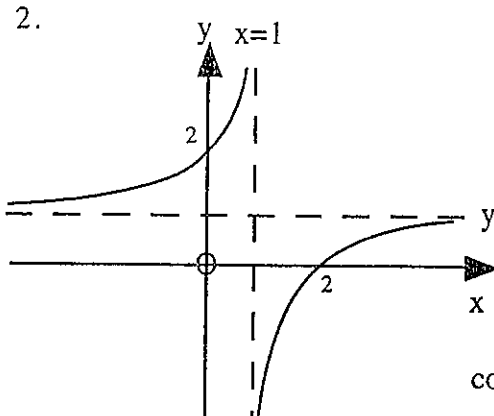Various sections on graphs.Page 45, 287, 290, 342, 348.

Answers

Exercise 1 Page 50

1. $x = 2$

2. $x = -3$ $x = 1$ 3. $x = -1$ $x = 3$ 4. $x = 2$

5. $x = -2$ $x = 2$ 6. $x = -2$ $x = 1$ 7. $x = 0$ 8. $x = 1$

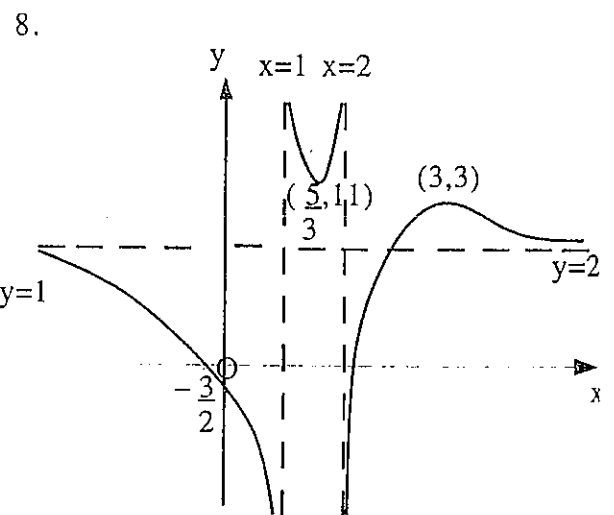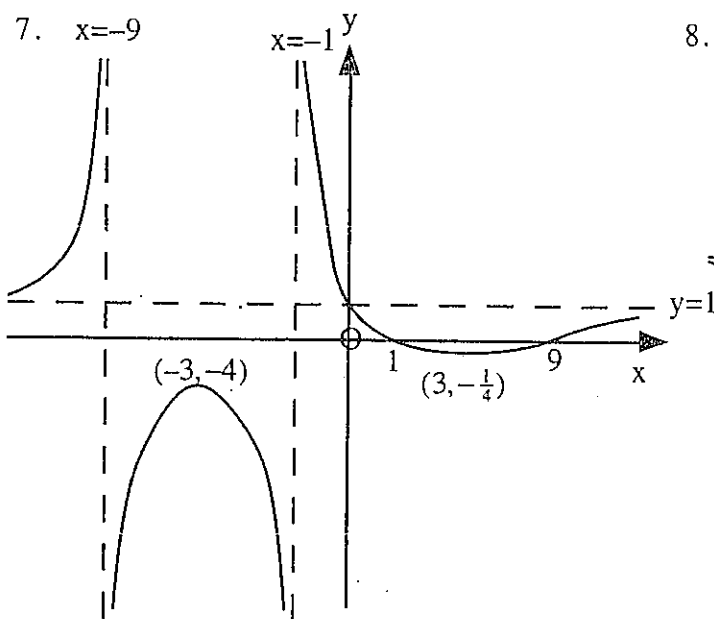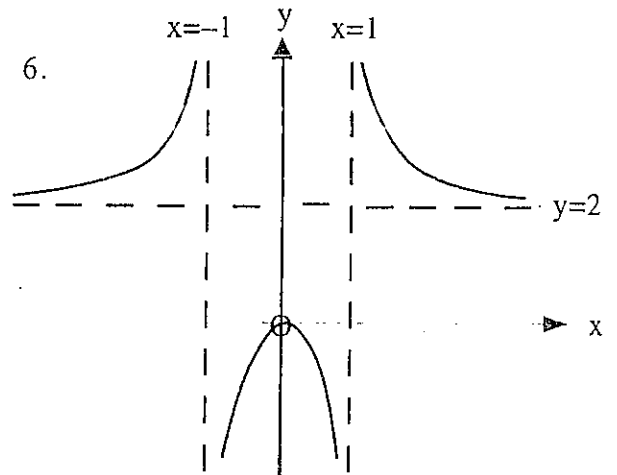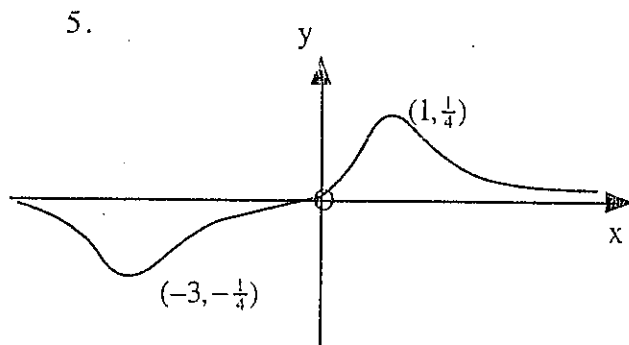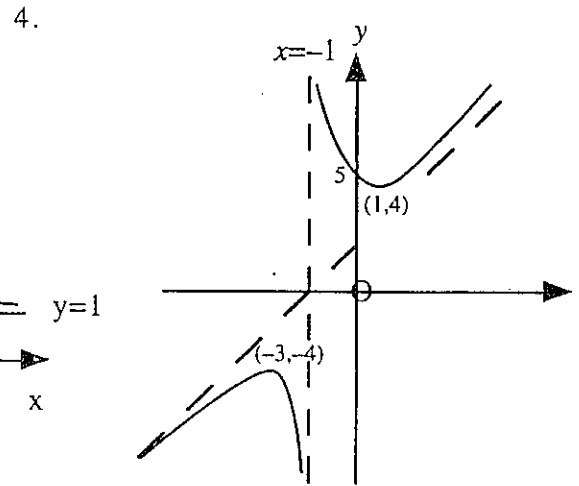9. None 10. None 11. $x = 1$ 12. $x = -1$ $x = 1$
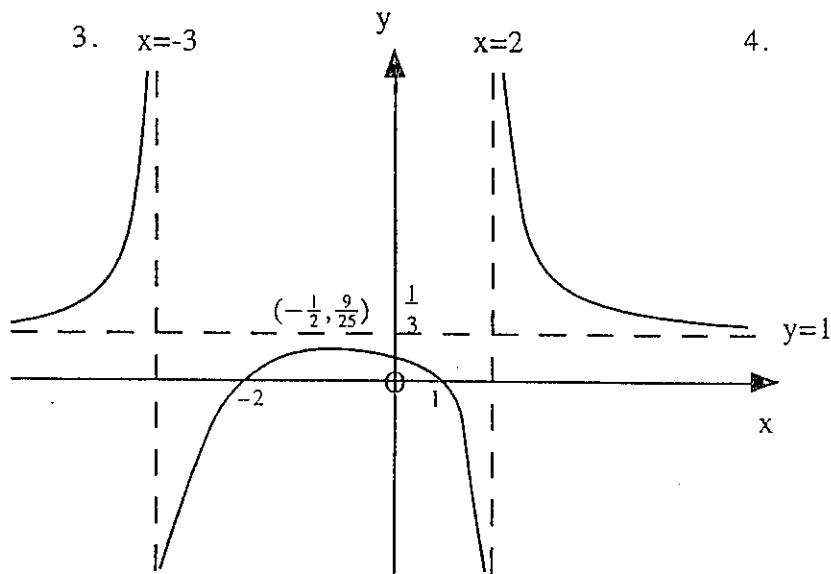
Exercise 2 Page 52

1. $y = 0$ 2. $y = 0$ 3. $y = 0$

4. $y = 1$ 5. $y = -1$ 6. $y = 1$

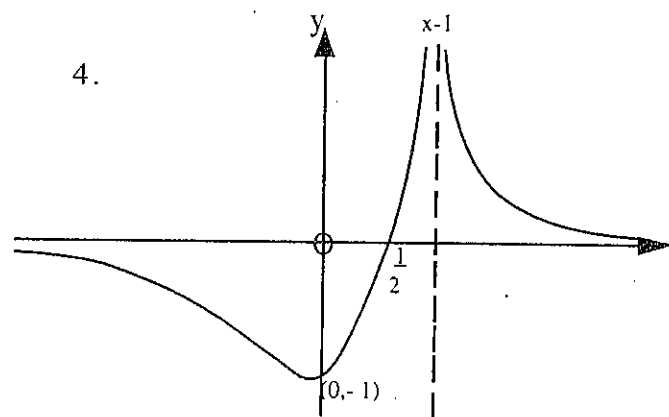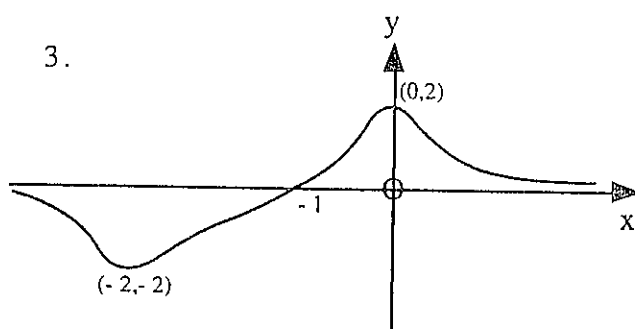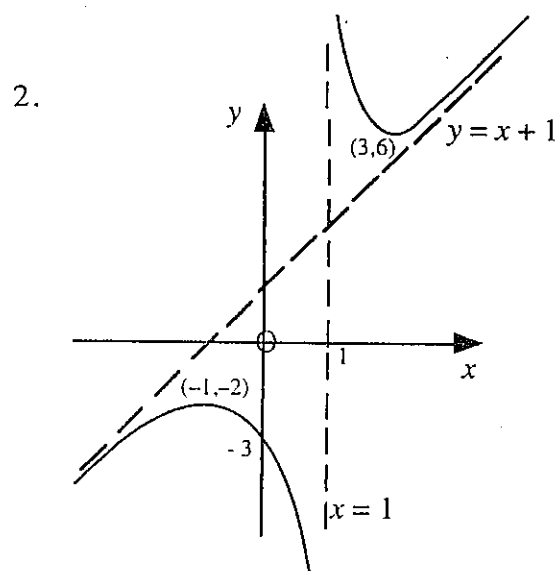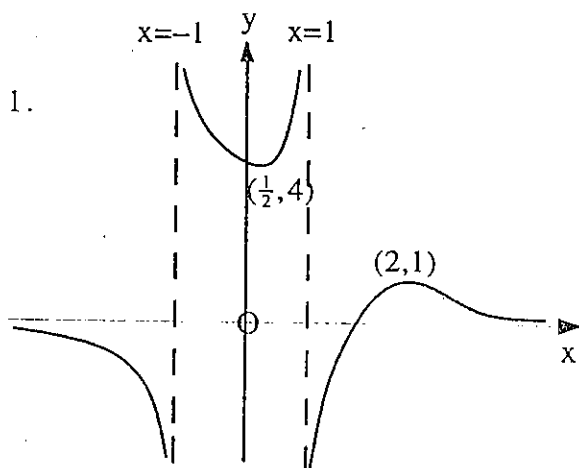7. $y = x - 3$ 8. $y = x + 1$ 9. $y = x$

10. $y = 0$ 11. $y = x + 1$ 12. $y = 2$

Exercise 3 Page 57

1.

2.

cont'd ....

3. $x=-3$ $x=2$

$(-\frac{1}{2}, \frac{9}{25})$ $\frac{1}{3}$

$y=1$

-2 1

4.

$x=-1$

5

$(1,4)$

$(-3,-4)$

5.

$(1, \frac{1}{4})$

$(-3, -\frac{1}{4})$

6. $x=-1$ $x=1$

$y=2$

7. $x=-9$ $x=-1$

$y=1$

$(-3,-4)$ 1 9

$(3, -\frac{1}{4})$

8. $x=1$ $x=2$

$(\frac{5}{3}, 1)$

$(3,3)$

$y=2$

$-\frac{3}{2}$

Exercise 4 Page 61

1.

2.

3.

4.

Exercise 5 Page 68

1. (a) $\frac{1}{2}x$ (b) $2 - x$ (c) $\frac{2}{x}$

   (d) $\log_2 x$ (e) $1 - x$ (f) $10^x + 2$

2. (a) $p/3$ (b) $p/6$ (c) $p/4$

   (d) $p/6$ (e) $-p/6$ (f) $p/6$

3. (a)

   (b)

3.   (c)

$(-1,5)$ $(0,4)$ $y = f(x) + 2$

$(2,3)$

(d)

$(-1,6)$ $(0,4)$ $y = 2f(x)$

$(2,1)$

$(-3,0)$

$y = -f(x)$

$(-3,0)$ $(0,-2)$ $(2,-1)$

$(-1,-3)$

$y = f(-x)$ $(1,3)$

$(0,2)$

(f)

$(-2,1)$

$(3,0)$

4.   (a)

$y = x + 2$ $y = |x + 2|$

2   2

-2 -2

(b)

5   $y = 5 - 2x$

$\frac{5}{2}$

5   $y = |5 - 2x|$

$\frac{5}{2}$

(c)

-1 3

-3

$y = x^2 - 2x - 3$

3

-1 3

$y = |x^2 - 2x -$

(d)

$y = 3x - x^2$

3

$y = |3x - x^2|$

3

page 72

(e)

$y = x^3 + 1$

$y = |x^3 + 1|$

(f)

$y = 1/x - 2$

$y = |1/x - 2|$
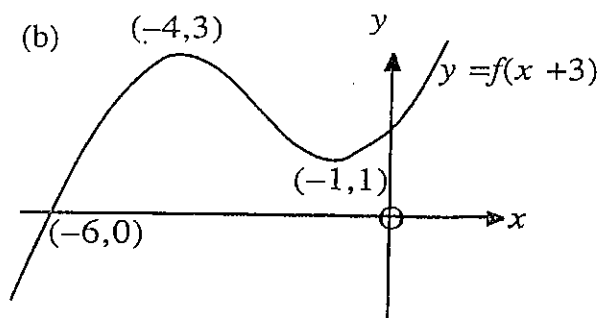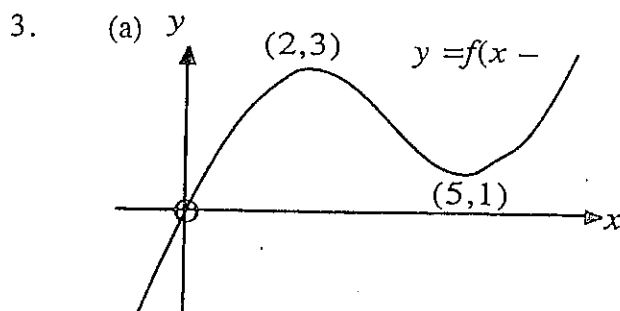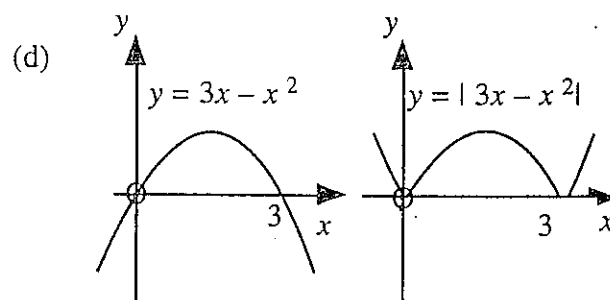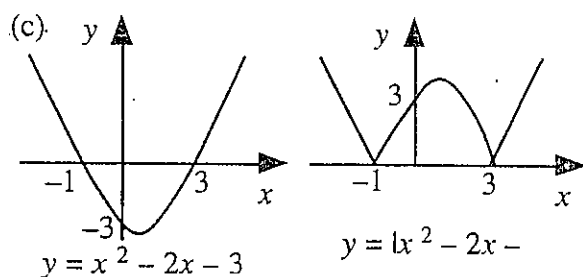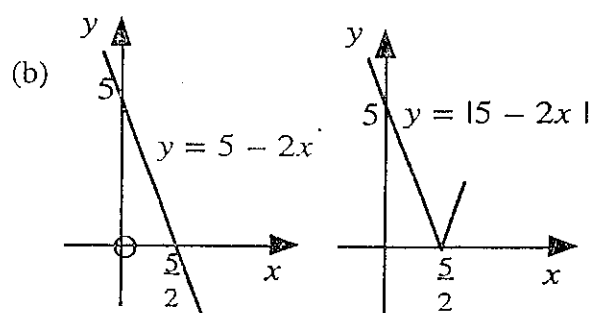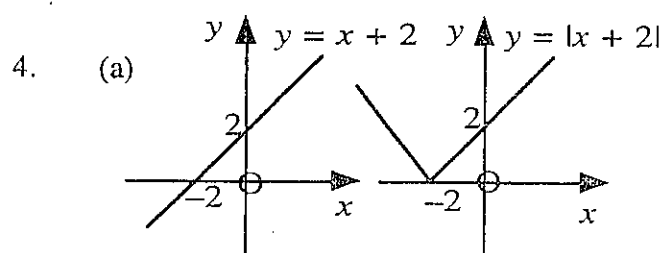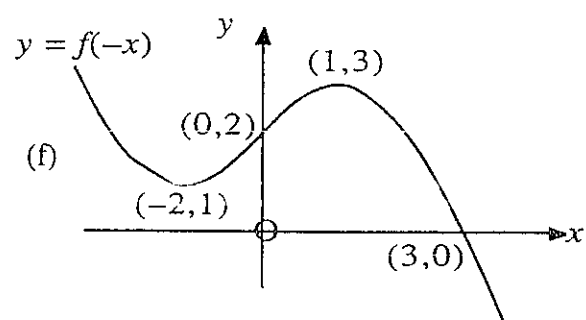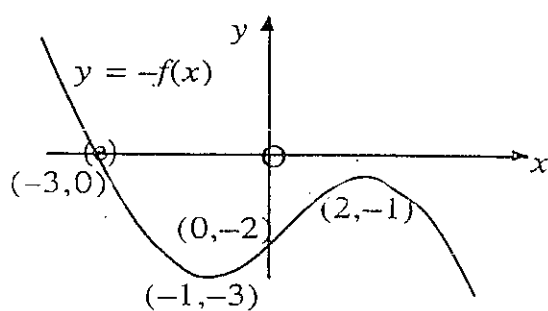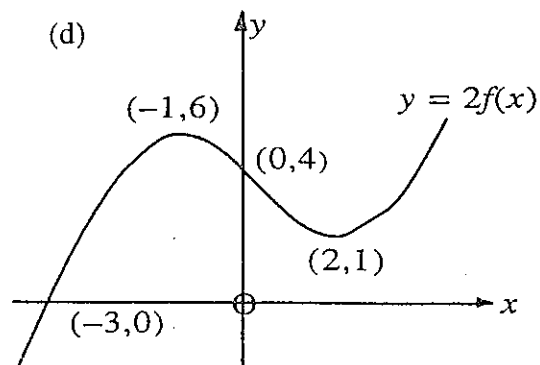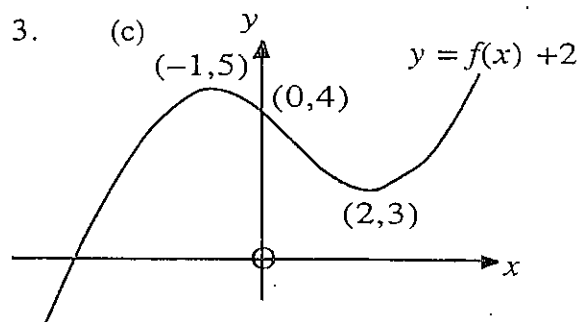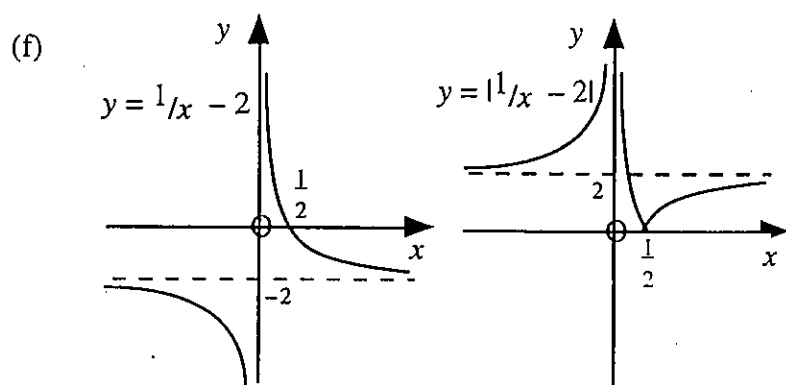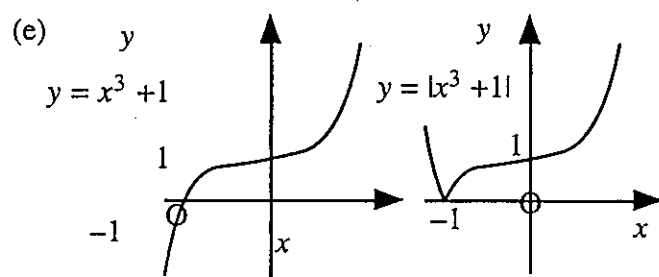
5. (a) neither   (b) even   (c) odd

    (d) odd   (e) even   (f) even

Advanced Higher Maths

Properties of Functions

2001

A function is defined by $f(x) = \dfrac{x^2 + 6x + 12}{x+2}$, $x \neq -2$.

    (a) Express $f(x)$ in the form $ax + b + \dfrac{b}{x+2}$ stating the values of a and b.

    (b) Write down an equation for each of the two asymptotes.

    (c) Show that $f(x)$ has two stationary points.

        Determine the coordinates and the nature of the stationary points.

    (d) Sketch the graph of f.

    (e) State the range of values of k such that the equation $f(x) = k$ has no solution.

(2, 2, 4, 1, 1 marks)

2002

Express $\dfrac{x^2}{(x+1)^2}$ in the form $A + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$, $(x \neq -1)$, stating the values of the constants A, B and C.

A curve is defined by $y = \dfrac{x^2}{(x+1)^2}$, $(x \neq -1)$.

    (i)   Write down equations for its asymptotes.

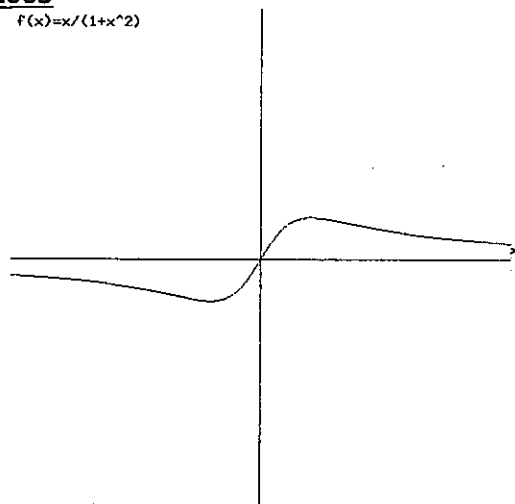   (ii)   Find the stationary point and justify its nature.

    (iii)   Sketch the curve showing clearly the features found in (i) and (ii).

(3, 2, 4, 2 marks)

2003

f(x)=x/(1+x^2)

The diagram shows the shape of $y = \dfrac{x}{1+x^2}$.

Obtain the stationary points of the graph.

Sketch the graph of $y = \left| \dfrac{x}{1+x^2} \right|$ and identify its
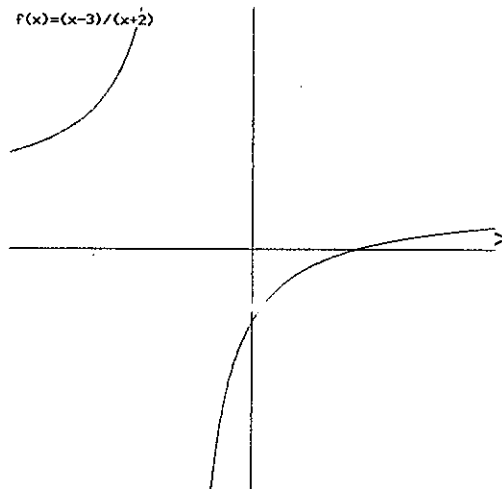
three critical points.

(4, 3 marks)

Advanced Higher Maths

2004

Determine whether the function $f(x) = x^4 \sin 2x$ is odd, even or neither.
Justify your answer.

(3 marks)

2004

The function f is defined by $f(x) = \dfrac{x-3}{x+2}$, $x \neq -2$, and the diagram shows part of its graph.
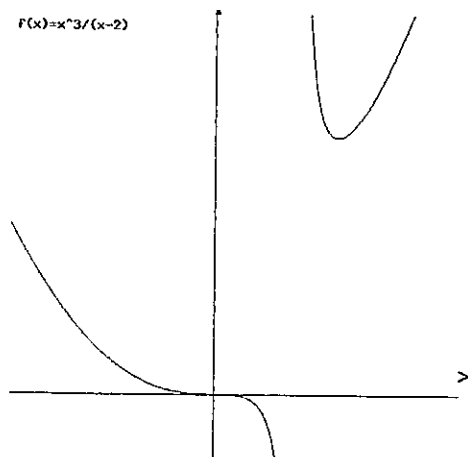

f(x)=(x−3)/(x+2)

(a) Obtain algebraically the asymptotes of the graph of f.

(b) Prove that f has no stationary values.

(c) Does the graph of f have any points of inflexion? Justify your answer.

(d) Sketch the graph of the inverse function f^{-1}.

State the asymptotes and domain of f^{-1}.
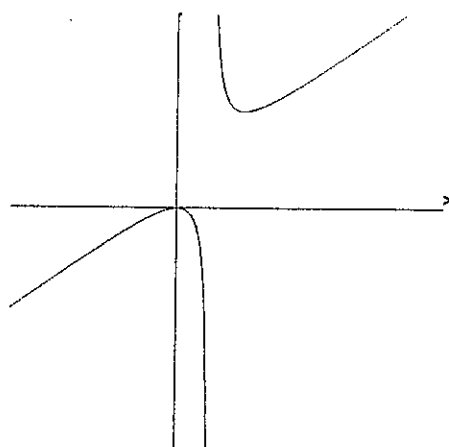
(3, 2, 2, 3 marks)

Advanced Higher Maths

2005

The diagram shows part of the graph of $y = \dfrac{x^3}{x-2}$, $x \neq 2$.

F(x)=x^3/(x-2)

(a) Write down the equation of the vertical asymptote.

(b) Find the coordinates of the stationary points of the graph of $y = \dfrac{x^3}{x-2}$.

(c) Write down the coordinates of the stationary points of the graph of $y = \left| \dfrac{x^3}{x-2} \right| + 1$.

(1, 4, 2 marks)

2006

The diagram shows part of the graph of a function f which satisfies the following conditions:
   (i) f is an even function;
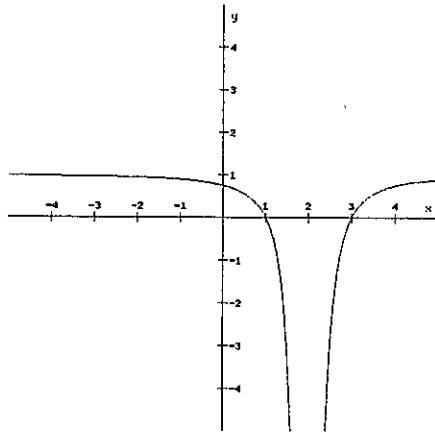   (ii)   two of the asymptotes of the graph $y = f(x)$ are $y = x$ and $x = 1$.

Copy the diagram and complete the graph. Write down equations for the other two asymptotes.

(3 marks)

Advanced Higher Maths

2008

Part of the graph $y = f(x)$ is shown below, where the dotted lines indicate asymptotes. Sketch the graph of $y = -f(x+1)$ showing its asymptotes. Write down the equations of the asymptotes.

(4 marks)

2009

The function $f(x)$ is defined by $f(x) = \dfrac{x^2 + 2x}{x^2 - 1}$ $(x \neq \pm 1)$.

Obtain equations for the asymptotes of the graph of $f(x)$.

Show that $f(x)$ is a strictly decreasing function.

Find the coordinates of the points where the graph of $f(x)$ crosses

(i) the x-axis and
(ii) the horizontal asymptote.

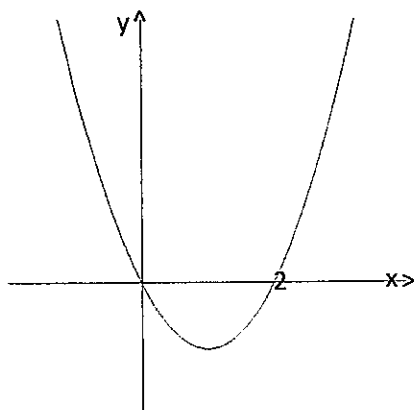Sketch the graph of $f(x)$, showing clearly all relevant features.

(3, 3, 2, 2 marks)

Advanced Higher Maths

2010

The diagram below shows part of the graph of a function $f(x)$.

State whether $f(x)$ is odd, even or neither. Fully justify your answer.

(3 marks)

2011
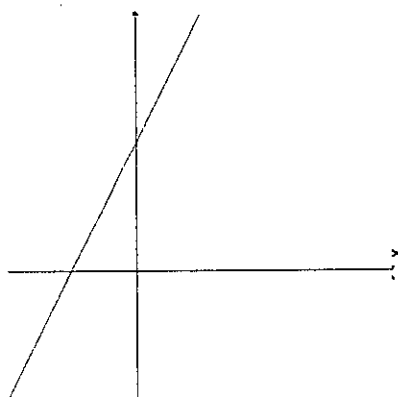
The diagram shows part of the graph of a function $f(x)$.

Sketch the graph of $\left|f^{-1}(x)\right|$ showing the points of intersection with the axes.

(4 marks)

2012

A function is defined by $f(x) = |x+2|$ for all x.

(a) Sketch the graph of the function for $-3 \le x \le 3$.

(b) On a separate diagram, sketch the graph of $f'(x)$.

(2, 2 marks)

Advanced Higher Maths
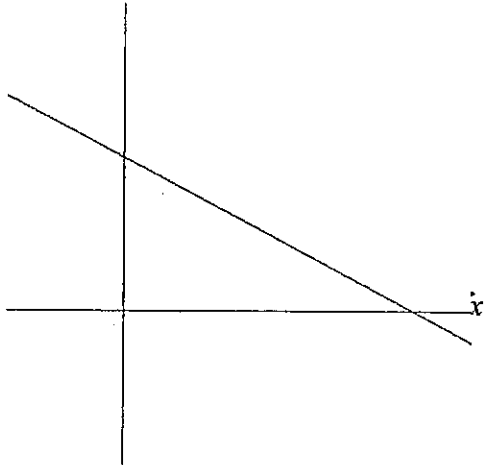
2013

Part of the straight line graph of a function $f(x)$ is shown.

(a) Sketch the graph of $f^{-1}(x)$, showing points of intersection with the axes.

(b) State the value of k for which $f(x)+k$ is an odd function.

(c) Find the value of h for which $\left| f(x+h) \right|$ is an even function.
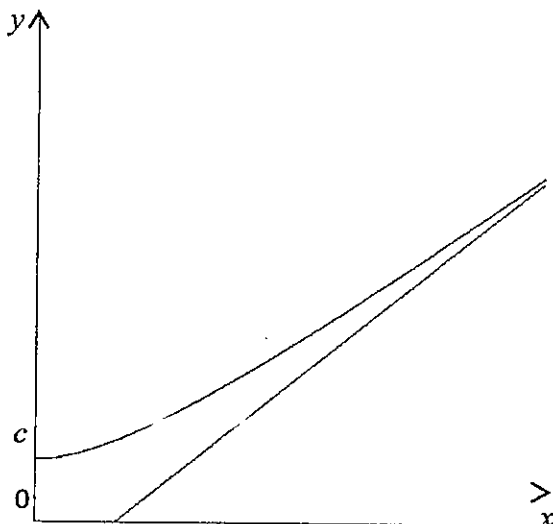
(2, 1, 2 marks)

Advanced Higher Maths

2014

The function $f(x)$ is defined for all $x \geq 0$.

The graph of $y = f(x)$ intersects the y-axis at $(0, c)$ where $0 < c < 5$.

The graph of the function and its asymptote, $y = x - 5$, are shown below.

(a) Copy the diagram above.

   On the same diagram, sketch the graph of $y = f^{-1}(x)$.

   Clearly show any points of intersection and any asymptotes.

(b) What is the equation of the asymptote of the graph $y = f(x+2)$?

(c) Why does your diagram show that the equation $x = f(f(x))$ has at least one solution?

(4, 1, 1 marks)

2015

For some function, f, define $g(x) = f(x) + f(-x)$ and
$$h(x) = f(x) - f(-x).$$

Show that $g(x)$ is an even function and that $h(x)$ is an odd function.

Hence show that $f(x)$ can be expressed as the sum of an even and an odd function.

(4 marks)

Functions

1   Critical and Stationary Points

A **critical point** exists where the derivative is zero or undefined.

Stationary points can be:

- maximum turning points;
- minimum turning points;
- horizontal points of inflection.

2   Derivative Tests

First derivative test

The first derivative test is the nature table method used in Higher.

Second derivative test

If a stationary point exists at $x = a$ then the second derivative test is to calculate $f''(a)$.

- If $f''(a) < 0$ then the point is a maximum turning point.
- If $f''(a) > 0$ then the point is a minimum turning point.

The second derivative test may not always work; if $f''(a) = 0$ then you should revert to the first derivative test.

Non-horizontal points of inflection

A **non-horizontal point of inflection** exists at $x = a$ when $f''(a) = 0$ but $f'(a) \neq 0$ and $f''(x)$ changes sign at $x = a$, i.e. the curve changes from being concave up to concave down or vice versa.

3 Curve Sketching

1. Identify x-axis and y-axis intercepts.
2. Identify turning points and their nature.
3. Consider the behaviour as $x \to \pm\infty$.
   This could be a horizontal asymptote or a slant asymptote (the equation of which is the polynomial part after algebraic long division).
4. Check where y is undefined (this means a vertical asymptote exists).
5. Sketch the curve showing asymptotes, turning points and x- and y-axis intercepts.

Transformations of curves

All transformations from Higher should be known.

The modulus function

To sketch the modulus of a function, take any part of the curve which is below the x-axis and reflect it in the x-axis.