Advanced Higher Maths

Unit 1.4

Integration 1

Outcome 3 – Integration

Integrate an expression requiring a standard result.

Standard Indefinite Integrals

$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}+c$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)}+c$
$\sin x$	$-\cos x+c$
$\cos x$	$\sin x+c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)+c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)+c$

All of the above were covered for the Higher course.
In addition the following trigonometrical identities were useful.

$$\sin^2 x=\frac{1}{2}(1-\cos 2x)$$
$$\cos^2 x=\frac{1}{2}(1+\cos 2x)$$

Exercise 1

Integrate the following functions :-

1. $f(x)=8x^3$
2. $f(x)=\frac{6}{x^3}$
3. $f(x)=\sqrt{x}$
4. $f(x)=\frac{1}{\sqrt{x}}$
5. $f(x)=\frac{3x^4+6}{x^2}$
6. $f(x)=\frac{1-3x}{\sqrt{x}}$
7. $f(x)=(3x+4)^4$
8. $f(x)=(1-2x)^4$
9. $f(x)=\frac{1}{(2x-3)^2}$
10. $f(x)=\frac{1}{\sqrt{(4x+1)}}$
11. $f(x)=\frac{3}{(3x+1)^{\frac{3}{2}}}$
12. $f(x)=\sin 3x$
13. $f(x)=\cos\left(\frac{1}{3}x\right)$
14. $f(x)=\sin^2 x$
15. $f(x)=\cos^2\left(\frac{1}{2}x\right)$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 171 Exercise 4.1:2 Questions 1, 2.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 296 Exercise 12A Questions 1, 2, 3 and
Page 373 Exercise 15G Questions 1, 3, 4, 8, 9, 10, 11, 12, 13, 14.

New Integrals

The exponential function $\int e^x dx$.

(a) If $f(x)=e^x$ then $f'(x)=e^x$ and so $\int e^x dx = e^x + c$

(b) If $f(x)=e^{2x}$ then $f'(x)=2e^{2x}$ and so $\int 2e^{2x} dx = e^{2x} + c$

ie $\int e^{2x} dx = \frac{1}{2}e^{2x} + c$

(c) If $f(x)=e^{1-2x}$ then $f'(x)=-2e^{1-2x}$ and so $\int -2e^{1-2x} dx = e^{1-2x} + c$

ie $\int e^{1-2x} dx = -\frac{1}{2}e^{1-2x} + c$

In General

$$\boxed{\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c}$$

The logarithmic function $\int \frac{1}{ax+b} dx$

(a) If $f(x)=\ln x$ then $f'(x)=\frac{1}{x}$ and so $\int \frac{1}{x} dx = \ln x + c$

(b) If $f(x)=\ln 2x$ then $f'(x)=\frac{2}{2x}$ and so $\int \frac{2}{2x} dx = \ln 2x + c$

ie $2\int \frac{1}{2x} dx = \ln 2x + c$

so $\int \frac{1}{2x} dx = \frac{1}{2}\ln 2x + c$

(c) If $f(x)=\ln(3x+2)$ then $f'(x)=\frac{3}{3x+2}$ and so $\int \frac{3}{3x+2} dx = \ln(3x+2) + c$

ie $3\int \frac{1}{3x+2} dx = \ln(3x+2) + c$

so $\int \frac{1}{3x+2} dx = \frac{1}{3}\ln(3x+2) + c$

In General

$$\boxed{\int \frac{1}{ax+b} dx = \frac{1}{a}\ln(ax+b) + c}$$

The integral of $\int \sec^2 x\,dx$

(a) If $f(x) = \tan x$ then $f'(x) = \sec^2 x$ and so $\int \sec^2 x\,dx = \tan x + c$

(b) If $f(x) = \tan 3x$ then $f'(x) = 3\sec^2 3x$ and so $\int 3\sec^2 3x\,dx = \tan 3x + c$

ie $3\int \sec^2 3x\,dx = \tan 3x + c$

so $\int \sec^2 3x\,dx = \frac{1}{3}\tan 3x + c$

In General

$$\boxed{\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c}$$

Exercise 2

Integrate the following functions :-

1. $f(x) = e^{5x}$	2. $f(x) = e^{-2x}$	3. $f(x) = 3e^{\frac{1}{2}x}$
4. $f(x) = \frac{1}{3x}$	5. $f(x) = \frac{1}{x+5}$	6. $f(x) = \frac{1}{2x-3}$
7. $f(x) = \left(e^x + e^{-x}\right)^2$	8. $f(x) = \frac{1+e^x}{e^x}$	9. $f(x) = e^{2x} + \frac{1}{e^{2x}}$
10. $f(x) = \frac{6}{3x+2}$	11. $f(x) = \frac{3}{1-2x}$	12. $f(x) = \frac{5}{6-7x}$
13. $f(x) = \sec^2 4x$	14. $f(x) = \sec^2(\pi + 2x)$	15. $f(x) = 3\sec^2 2x$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 172 Exercise 4.1:2 Questions 15, 18.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 480 Exercise 19A Questions 16, 17, 18, 19, 22, 23. and
Page 487 Exercise 19B Questions 26, 27, 28, 34, 35, 36.

Definite Integrals

1. $\int_a^b f(x)\,dx = F(a) - F(b)$ *where* $F'(x) = f(x)$

eg. $\int_1^3 (x^2+1)dx = \left[\frac{x^3}{3}+x\right]_1^3 = \left(\frac{3^3}{3}+3\right)-\left(\frac{1^3}{3}+1\right) = 12-\frac{4}{3} = \frac{32}{3}$ or $10\frac{2}{3}$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,\mathrm{dx}$

eg. $\int_1^3 (x^2+1)dx = \left[\frac{x^3}{3}+x\right]_1^3 = \left(\frac{3^3}{3}+3\right)-\left(\frac{1^3}{3}+1\right) = 12-\frac{4}{3} = 10\frac{2}{3}$

and $-\int_3^1 (x^2+1)dx = -\left[\frac{x^3}{3}+x\right]_3^1 = -\left[\left(\frac{1^3}{3}+1\right)-\left(\frac{3^3}{3}+3\right)\right] = -\left[\frac{4}{3}-12\right] = -\left[-\frac{32}{3}\right] = \frac{32}{3} = 10\frac{2}{3}$

Examples

1. $\int_{-3}^{-2} \frac{1}{x^2}\,dx = \int_{-3}^{-2} x^{-2}\,dx = \left[-x^{-1}\right]_{-3}^{-2} = \left[-\frac{1}{x}\right]_{-3}^{-2} = \left(-\frac{1}{-2}\right)-\left(-\frac{1}{-3}\right) = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$

2. $\int_1^5 \frac{1}{\sqrt{(2x-1)}}dx = \int_1^5 (2x-1)^{-\frac{1}{2}}dx = \left[\frac{(2x-1)^{\frac{1}{2}}}{2\times\frac{1}{2}}\right]_1^5 = \left[(2x-1)^{\frac{1}{2}}\right]_1^5 = 9^{\frac{1}{2}}-1^{\frac{1}{2}} = 3-1 = 2$

3. $\int_0^{\frac{1}{2}\pi} \sin 2x dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{1}{2}\pi} = \left(-\frac{1}{2}\cos\pi\right)-\left(-\frac{1}{2}\cos 0\right) = \frac{1}{2}+\frac{1}{2} = 1$

4. $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sec^2\frac{1}{2}x\,\mathrm{dx} = \left[2\tan\frac{1}{2}x\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = 2\tan\frac{1}{4}\pi - 2\tan\left(-\frac{1}{4}\pi\right) = 2\times 1 - 2\times(-1) = 4$

5. $\int_0^2 e^{-3x}dx = \left[-\frac{1}{3}e^{-3x}\right]_0^2 = \left(-\frac{1}{3}e^{-6}\right)-\left(-\frac{1}{3}e^{0}\right) = \left(-\frac{1}{3}e^{-6}\right)-\left(-\frac{1}{3}\right) = \frac{1}{3}\left(1-e^{-6}\right)$

6. $\int_4^5 \frac{2}{x-3}dx = \left[2\ln(x-3)\right]_4^5 = 2\ln 2 - 2\ln 1 = 2\ln 2 = \ln 4$

Exercise 3

Integrate the following functions :-

1.	$\int_1^2 \left(\frac{2}{x^2} - \frac{5}{x^4}\right) dx$	2.	$\int_{\frac{1}{4}}^1 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$	3.	$\int_1^4 \frac{2x^2+3}{\sqrt{x}} dx$
4.	$\int_1^6 \sqrt{(x+3)} dx$	5.	$\int_0^1 \frac{1}{(4+5x)^2} dx$	6.	$\int_3^{12} (x-4)^{\frac{1}{3}} dx$
7.	$\int_0^{\frac{\pi}{4}} \cos 2x dx$	8.	$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec}^2 x dx$	9.	$\int_0^{2\pi} \sin^2 x dx$
10.	$\int_0^2 e^{-3x} dx$	11.	$\int_0^1 e^{1-x} dx$	12.	$\int_0^1 e^{\frac{x}{2}} dx$
13.	$\int_5^9 \frac{1}{x-3} dx$	14.	$\int_0^1 \frac{1}{3x+2} dx$	15.	$\int_{-4}^0 \frac{1}{1-2x} dx$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 176 Exercise 4.1:3 Questions 1, 2, 3, 4.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 303 Exercise 12B Question 1. and Page 374 Exercise 15G Questions 36, 37, 38, 41.

Integration by Substitution

It is sometimes difficult to reduce an integral to one of the Standard integrals. Such integrals may be made simpler by changing the variable by means of a substitution of a new variable.

Examples (where the simple substitution is not given)

1. $\int x^4\left(1+2x^5\right)^3 dx$ $\quad\left(\text{note that } \frac{d}{dx}\left(1+2x^5\right)=10x^4\right)$

Let $u=1+2x^5$ then $\frac{du}{dx}=10x^4$ and so $x^4\,dx=\frac{1}{10}\,du$

The integral now becomes

$$\int\frac{1}{10}u^3\,du=\frac{1}{10}\times\frac{1}{4}\times u^4+c$$
$$=\frac{1}{40}\left(1+2x^5\right)^4+c$$

2. $\int 2xe^{x^2}dx$ $\quad\left(\text{note that } \frac{d}{dx}\left(x^2\right)=2x\right)$

Let $u=x^2$ then $\frac{du}{dx}=2x$ and so $2xdx=du$

The integral then becomes

$$\int e^u\,du = e^u+c$$
$$= e^{x^2}+c$$

3. $\int sin^2xcosx\,dx$ $\quad\left(\text{note that } \frac{d}{dx}(sinx)=cosx\right)$

Let $u=sinx$ then $\frac{du}{dx}=cosx$ and so $cosxdx=du$

The integral then becomes

$$\int u^2\,du=\frac{1}{3}u^3+c$$
$$=\frac{1}{3}sin^3x+c$$

4. $\int\frac{lnx}{x}dx$ $\quad\left(\text{note that } \frac{d}{dx}(lnx)=\frac{1}{x}\right)$

Let $u=lnx$ then $\frac{du}{dx}=\frac{1}{x}$ and so $\frac{1}{x}dx=du$

The integral then becomes

$$\int u\,du=\frac{1}{2}u^2+c$$
$$=\frac{1}{2}(lnx)^2+c$$

5. $\int x^2\left(2x^3+3\right)^5 dx$

Let $u = 2x^3 + 3$ then $\frac{du}{dx} = 6x^2$ and so $6x^2dx = du$ and $x^2\,dx = \frac{1}{6}\,du$

The integral then becomes

$$\int \frac{1}{6}u^5\,du = \frac{1}{36}u^6 + c$$
$$= \frac{1}{36}\left(2x^3 + 5\right)^6 + c$$

6. $\int 2x(x+2)^5\,dx$

Let u = x + 2 then $\frac{du}{dx} = 1$ and so $dx = du$ and from u = x + 2 then x = u − 2

The integral then becomes

$$\int 2(u-2)u^5\,du = \int\left(2u^6 - 4u^5\right)du = \frac{2}{7}u^7 - \frac{2}{3}u^6 + c$$
$$= \frac{2}{7}(x+2)^7 - \frac{2}{3}(x+2)^6 + c$$

7. $\int \frac{x^4}{\left(x^5+1\right)^3}\,dx$

Let $u = x^5 + 1$ then $\frac{du}{dx} = 5x^4$ and so $x^4dx = \frac{1}{5}du$

The integral then becomes

$$\int \frac{1}{5u^3}\,du = \frac{1}{5}\int u^{-3}du = \frac{1}{5}\times -\frac{1}{2}u^{-2} + c = -\frac{1}{10u^2} + c$$
$$= -\frac{1}{10\left(x^5+1\right)^2} + c$$

8. $\int \frac{2x}{x^2+4}\,dx$

Let $u = x^2 + 4$ then $\frac{du}{dx} = 2x$ and so $2xdx = du$

The integral then becomes

$$\int \frac{du}{u} = \ln u + c$$
$$= \ln\left(x^2 + 4\right) + c$$

9. $\int \frac{x}{x^2+5}\,dx$

Let $u = x^2 + 5$ then $\frac{du}{dx} = 2x$ and so $2xdx = du$ and $xdx = \frac{1}{2}du$

The integral then becomes

$$\int \frac{1}{2u}\,du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u + c$$
$$= \frac{1}{2}\ln\left(x^2 + 5\right) + c$$

contd....

10. $\int tanx dx = \int \frac{sinx}{cosx} dx$

Let $u = cosx$ then $\frac{du}{dx} = -sinx$ and so $sinx dx = -du$

The integral then becomes

$$\int -\frac{1}{u} du = -lnu + c$$
$$= -ln(cosx) + c$$

Questions 9, 10, and 11 demonstrated the special integral of the form

$$\boxed{\int \frac{f'(x)}{f(x)} dx = ln[f(x)] + c}$$

Simple substitution will not be given whereas more difficult ones will be given.

Exercise 4

Integrate the following functions (no substitution given) :-

1. $\int x(x^2 - 3)^5 dx$
2. $\int x^2(x^3 - 1)^2 dx$
3. $\int x\sqrt{(1 - x^2)} dx$
4. $\int cosx sin^4 x dx$
5. $\int \frac{x}{(1 - x^2)^3} dx$
6. $\int \frac{x^3}{1 + x^4} dx$
7. $\int \frac{sinx}{cos^3 x} dx$
8. $\int \frac{cosx}{sin^6 x} dx$
9. $\int \frac{e^x}{3e^x - 1} dx$
10. $\int \frac{sec^2 x}{tanx} dx$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 189 Exercise 4.2:2 Questions 1,3,5,7.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 320 Exercise 13B Questions 1 – 20.

Examples (where the substitution will be given)

1. $\int x\sqrt{(3x - 2)}\ dx$ given $u = 3x - 2$

Let $u = 3x - 2$ then $\frac{du}{dx} = 3$ and so $dx = \frac{1}{3} du$

and from $u = 3x - 2$ then $x = \frac{1}{3}(u + 2)$

The integral then becomes

$$\int \frac{1}{3}(u + 2)\sqrt{u}\frac{1}{3}\ du = \int \frac{1}{9}\left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du = \frac{1}{9}\left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + c$$
$$= \frac{2}{45}(3x - 2)^{\frac{5}{2}} + \frac{2}{27}(3x - 2)^{\frac{3}{2}} + c$$

contd....

2. $\int x(x+2)^5 dx$ given $u = x + 2$

Let $u = x + 2$ then $\frac{du}{dx} = 1$ and so $du = dx$

and from $u = x + 2$ then $x = u - 2$

The integral then becomes

$$\int (u-2)u^5 du = \int \left(u^6 - 2u^5\right) du = \frac{1}{7}u^7 - \frac{1}{3}u^6 + c = \frac{1}{7}(x+2)^7 - \frac{1}{3}(x+2)^6 + c$$

3. $\int \frac{x}{\sqrt{(x+4)}} dx$ given $u = x + 4$

Let $u = x + 4$ then $\frac{du}{dx} = 1$ and so $du = dx$

and from $u = x + 4$ then $x = u - 4$

The integral then becomes

$$\int \frac{u-4}{\sqrt{u}} du = \int u^{\frac{1}{2}} - 4u^{-\frac{1}{2}} du = \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + c = \frac{2}{3}(x+4)^{\frac{3}{2}} - 2(x+4)^{\frac{1}{2}} + c$$

Exercise 5

Integrate the following functions (substitution given) :-

1. $\int 9x(3x+2)^3 dx$ where $u = 3x + 2$
2. $\int 7x(2x+3)^5 dx$ where $u = 2x + 3$
3. $\int 3x\sqrt{\left(1+x^2\right)} dx$ where $u = 1 + x^2$
4. $\int \frac{3x}{\sqrt{(2x+3)}} dx$ where $u = 2x + 3$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 186 Exercise 4.2:1 Question 1. and Page 189 Exercise 4.2:2 Question 1.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 500 Exercise 20A Questions 3 – 9.

<u>Examples</u> (where the substitution will be given – Special substitutions) -

In integrals which contain the expression $\sqrt{(a^2 - x^2)}$, one of the following could be given :

either $u = \sqrt{(a^2 - x^2)}$ or $x = a\sin\theta$

1. $\int \sqrt{\left(1 - x^2\right)} dx$ given $x = \sin\theta$

From $x = \sin\theta$, $\frac{dx}{d\theta} = \cos\theta$, $dx = \cos\theta \, d\theta$

and $\sqrt{(1 - x^2)} = \sqrt{(1 - \sin^2\theta)} = \sqrt{(\cos^2\theta)} = \cos\theta$

The integral becomes

$$\int \cos\theta\cos\theta d\theta = \int \cos^2\theta d\theta = \int \frac{1}{2}(1 + \cos 2\theta) d\theta = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c$$

contd....

We must return to the original variable.

Since $\sin\theta = x$ then $\theta = \sin^{-1}x$

$\cos\theta = \sqrt{(1-x^2)}$ and $\sin 2\theta = 2\sin\theta\cos\theta = 2x\sqrt{(1-x^2)}$

$$\int\sqrt{(1-x^2)}\,dx = \frac{1}{2}\sin^{-1}x + \frac{1}{4}2x\sqrt{(1-x^2)} + c$$
$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{(1-x^2)} + c$$

2. $\displaystyle\int\frac{x^2}{\sqrt{(9-x^2)}}dx$ given $x = 3\sin\theta$

From $x = 3\sin\theta$, $\dfrac{dx}{d\theta} = 3\cos\theta$, $dx = 3\cos\theta\,d\theta$

and $\sqrt{(9-x^2)} = \sqrt{(9-9\sin^2\theta)} = \sqrt{(9\cos^2\theta)} = 3\cos\theta$

The integral becomes

$$\int\frac{9\sin^2\theta.3\cos\theta}{3\cos\theta}d\theta = \int 9\sin^2\theta\,d\theta = \int 9.\frac{1}{2}(1-\cos 2\theta)d\theta = \int\left(\frac{9}{2}-\frac{9}{2}\cos 2\theta\right)d\theta$$
$$= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + c$$

We must return to the original variable.

Since $\sin\theta = \dfrac{x}{3}$ then $\theta = \sin^{-1}\left(\dfrac{x}{3}\right)$

$$\cos\theta = \sqrt{(1-\sin^2 x)} = \sqrt{9-\left(\frac{x}{3}\right)^2} = \sqrt{9-\frac{x^2}{9}} = \sqrt{\frac{9-x^2}{3}} = \frac{\sqrt{9-x^2}}{3}$$

and $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\dfrac{x}{3}\right)\dfrac{\sqrt{9-x^2}}{3} = \dfrac{2}{9}x\sqrt{(9-x^2)}$

$$\int\frac{x^2}{\sqrt{(9-x^2)}}dx = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{(9-x^2)} + c$$

Exercise 6

Integrate the following functions (substitution given) :-

1. $\displaystyle\int\frac{x}{\sqrt{(1-x^2)}}dx$ given $x = \sin\theta$

2. $\displaystyle\int\sqrt{(4-x^2)}\,dx$ given $x = 2\sin\theta$

3. $\displaystyle\int\frac{x}{\sqrt{(9-x^2)}}dx$ given $x = 3\sin\theta$

4. $\displaystyle\int\frac{x^2}{\sqrt{(4-x^2)}}dx$ given $x = 2\sin\theta$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 193 Exercise 4.2:3 Question 1,2,3,5,7.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 500 Exercise 20A Questions 10 – 16.

Substitution and definite integrals

Assuming the function is continuous over the interval of integration, then exchanging the limits for x by the corresponding limits for u will save you having to substitute back after the integration process.

Example Evaluate $\int_1^2 (2x+4)(x^2+4x)^3\,dx$.

Let $u = x^2 + 4x$. Then $du = (2x+4)\,dx$; also, when $x = 2$, $u = 12$; when $x = 1$, $u = 5$.
Substituting gives:

$$\int_5^{12} u^3\,du = \left[\frac{u^4}{4}\right]_5^{12} = 5027.75$$

EXERCISE 5A

Find the following definte integrals using the given substitution.

1 $\int_0^1 x^3(2+x^4)^5\,dx,\ u = x^4 + 2$

2 $\int_4^9 \frac{x+1}{x^2+2x-7}\,dx,\ u = x^2 + 2x - 7$

3 $\int_0^4 \frac{\sqrt{x}}{2+\sqrt{x}}\,dx,\ u = \sqrt{x} + 2$

4 $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx,\ u = 1 - x^2$

5 $\int_0^1 x\sqrt{2x^2+3}\,dx,\ u = 2x^2 + 3$

6 $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\,dx,\ u = 1 + \cos x$

7 $\int_0^{\frac{\pi}{2}} \frac{1-2\sin x}{x+2\cos x}\,dx,\ u = x + 2\cos x$

8 $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot x\,dx,\ u = \sin x$

9 $\int_0^{\frac{\pi}{3}} \sin x\cos^4 x\,dx,\ u = \cos x$

10 $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x}\,dx,\ u = \cos x$

11 $\int_e^{e^2} \frac{1}{x\ln x}\,dx,\ t = \ln x$

12 $\int_1^2 \frac{e^{2x}}{(e^{2x}-1)^2}\,dx,\ t = e^{2x} - 1$

EXERCISE 5B

1 Find $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$, using the substitution $u = \cos x$ and remembering that $\sin^2 x + \cos^2 x = 1$.

2 Find $\int_0^{\frac{\pi}{4}} \sin^3 x\cos^2 x\,dx$ making the substitution $u = \cos x$.

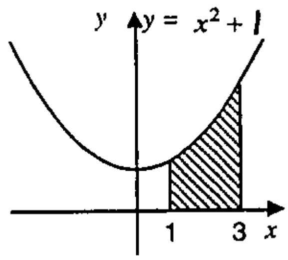
ANSWERS FOR EX5A + EX5B AT
THE END OF THE BOOKLET

The Area enclosed by a curve and the x axis

The Area enclosed by a curve and the x axis has been already met at Higher level.
$A = \int_a^b f(x)dx$ or $\int_a^b ydx = F(b) - F(a)$ where $F(x)$ is the integral of $f(x)$ or y and $F(b)$ and $F(a)$ are the values of $F(x)$ at $x = b$ and $x = a$.

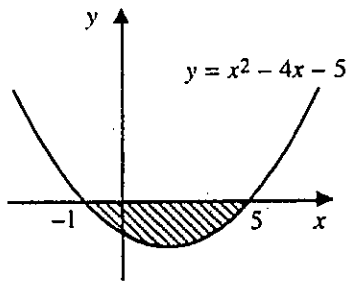
Examples

1. Find the area enclosed by the curve $f(x) = x^2 + 1$ between $x = 1$ and $x = 3$.
The area is above the x-axis



$$\int_1^3 ydx = \int_1^3 (x^2+1)dx = \left[\frac{x^3}{3}+1\right]_1^3$$
$$= (9 + 1) - (1/3 + 1)$$
$$\Rightarrow \text{Area} = \frac{26}{3}$$

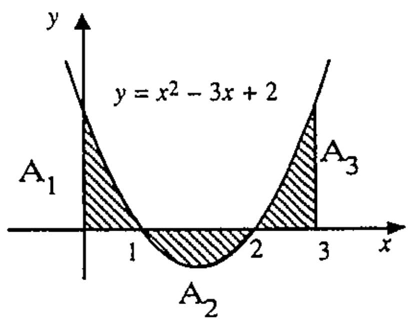
2. Find the area enclosed by the curve $f(x) = x^2 - 4x - 5$ and the x-axis.
The area is below the x-axis.



$$\int_{-1}^5 ydx = \int_{-1}^5 (x^2 - 4x - 5)dx$$
$$= \left[\frac{x^3}{3} - 2x^2 - 5x\right]_{-1}^5$$
$$= (125/3 - 50 - 25) - (-1/3 - 2 - 5)$$
$$= (-100/3) - (-22/3)$$
$$= -78/3$$

The negative indicates that the area enclosed is below the x-axis as shown.
Area = $78/3$

3. Find the area enclosed by the curve $f(x) = x^2 - 3x + 2$ and the x-axis, from $x = 0$ to $x = 3$.
The area lies partly above and partly below the x-axis.



$$A_1 = \int_0^1 ydx = \int_0^1 (x^2 - 3x + 2)dx$$
$$= \left[\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right]_0^1 = 5/6$$
$$A_2 = \int_1^2 f(x)dx = \int_1^2 (x^2 - 3x + 2)dx$$
$$= \left[\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right]_1^2 = -1/6$$
$$A_3 = \int_2^3 f(x)dx = \int_2^3 (x^2 - 3x + 2)dx$$

$\Rightarrow$ The total area $= 5/6 + 1/6 + 5/6 = 1\,5/6$

contd....

Exercise 7

1. Find the area enclosed by the following curves and the x-axis between the lines given.

   (a) $y = 3x^2 + 2, \quad x = 0, x = 2$

   (b) $y = x^3 - x, \quad x = 1, x = 2$

2. Find the area enclosed by the following curves and the x-axis.
   (A rough sketch could help)

   (a) $y = 6 + x - x^2$

   (b) $y = x(x - 2)(x - 3)$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 176 Exercise 4.1:3 Question 5,6,7,8,9,25.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 304 Exercise 12B Questions 2 - 14.

The Area enclosed between two curves

The Area enclosed between two curves has been already met at Higher level.

$A = \int_a^b [f(x) - g(x)]dx$ where the curves $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$

Note Provided the graph of $f(x)$ is above the graph of $g(x)$, their positions relative to the x-axis does not matter.

Examples

1. Find the area enclosed between the curve $y = x(4 - x)$ and the straight line $y = x$.

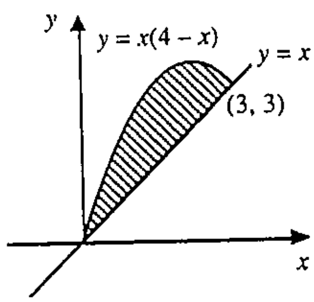
The curve and straight line meet where

$$x = x(4 - x)$$

i.e. $x^2 - 3x = 0$

$x(x - 3) = 0$

i.e. $x = 0$ and $x = 3$



$$A = \int_0^3 [x(4 - x) - x]dx = \int_0^3 (3x - x^2)dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_0^3$$

$$= (\frac{27}{2} - 9) - 0 \quad = \quad \frac{9}{2}$$

contd....

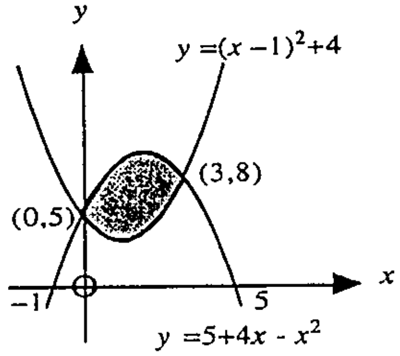
2. Find the area enclosed between the curves $y = (x - 1)^2 + 4$ and $y = 5 + 4x - x^2$.

The curves meet where

$(x - 1)^2 + 4 = 5 + 4x - x^2$

i.e. $2x^2 - 6x = 0 \Rightarrow 2x(x - 3) = 0$

i.e. $x = 0$ and $x = 3$



$$A = \int_0^3 \left[\left(5 + 4x - x^2\right) - \left(x^2 - 2x + 5\right)\right]dx$$

$$= \int_0^3 \left(6x - 2x^2\right)dx = \left[3x^2 - \frac{2}{3}x^3\right]_0^3$$

$$= (27 - 18) - 0$$

$$= 9$$

Exercise 8

Find the areas enclosed by the following curves.

1. $y = x(10 - x)$ and $y = 4x$.
2. $y = 4x - x^2$ and $y = x^2 - 4x + 6$.
3. $y = 2\sqrt{x}$ and $y = \dfrac{x^2}{4}$
4. $y = x^3 + x^2 - 5x$ and $y = x^2 - x$.
5. $y = \sin x$ and $y = \cos x$
6. $y^2 = 4ax$ and $x^2 = 4ay$ (both are parabolas)

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 176 Exercise 4.1:3 Questions 48,50,51,52.

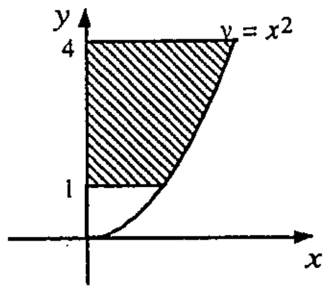
Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 303 Exercise 12B Questions 16 – 22.

The Area enclosed by a curve and the y-axis

The Area enclosed by a curve and the y-axis is given by
$A = \int_a^b f(y)dy$ or $\int_a^b x dy = F(b) - F(a)$ where $F(y)$ is the integral of $f(y)$ or y and $F(b)$ and $F(a)$ are the values of $F(y)$ at $y = b$ and $y = a$.

Examples

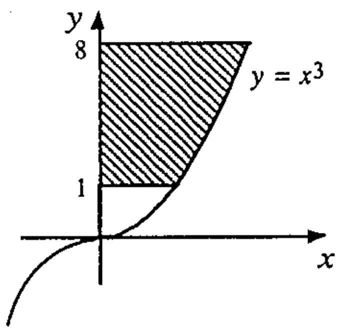
1. Find the area enclosed by the curve $y = x^2$ between $y = 1$ and $y = 4$, $x \geq 0$.



From $y = x^2$, $x = \sqrt{y}$

$$A = \int_1^4 y^{\frac{1}{2}} dy = \left[\frac{2}{3} y^{\frac{3}{2}}\right]$$
$$= \left(\frac{2}{3}(4)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}}\right)$$
$$= {}^{16}/_3 - {}^{2}/_3 \quad = 4^{2}/_3$$

2. Find the area enclosed by the curve $y = x^3$ between $y = 1$ and $y = 8$.
From $y = x^3$, $x = \sqrt[3]{y}$



$$A = \int_1^8 y^{\frac{1}{3}} dy = \left[\frac{3}{4} y^{\frac{4}{3}}\right]_1^8 = \left(\frac{3}{4}(8)^{\frac{4}{3}}\right) - \left(\frac{3}{4}(1)^{\frac{4}{3}}\right)$$
$$= 12 - {}^{3}/_4 \quad = 11^{1}/_4$$

Exercise 9

Find the area enclosed by the following curves and the y-axis between the lines given.

1. $x = y^2$, $y = 3$.
2. $y = x^3$, $y = 1$, $y = 8$.
3. $x = \frac{1}{\sqrt{y}}$, $y = 2$, $y = 3$.
4. $y^2 = 1 - x$, $y = 0$, $y = 1$.
5. $y = {}^{1}/x^3$, $y = 8$, $y = 27$.
6. $y = \ln x$, $y = 2$, $y = 5$

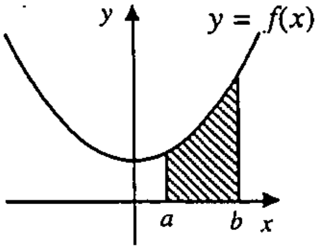
Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 181 Exercise 4.1:4 Questions 16(Area only).

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 304 Exercise 12B Question 15.

The Volume of Revolution

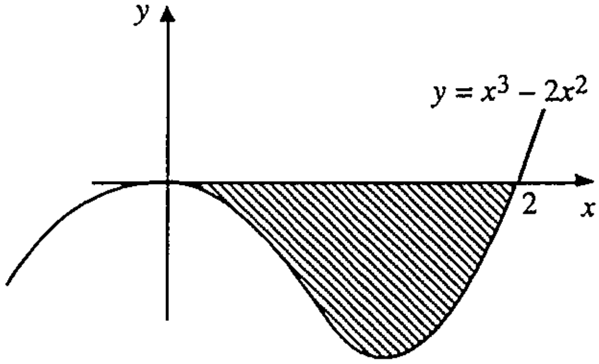
The volume of revolution is formed when the area bounded by a curve $y = f(x)$, the x - axis, x = a and x = b is rotated completely about the x – axis.



$$V = \int_a^b \pi y^2 dx$$

Example

Find the volume generated, by rotating about the x – axis, the area enclosed by the curve $y = x^3 - 2x^2$ and the x – axis.



The curve crosses the x axis where

$$x^3 - 2x^2 = 0$$

$$x^2(x - 2) = 0 \Rightarrow x = 0 \text{ and } x = 2$$

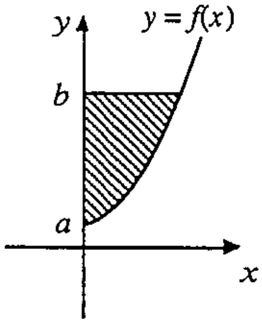
$$\therefore \quad V = \int_0^2 \pi(x^3 - 2x^2)^2 dx$$

$$= \int_0^2 \pi(x^6 - 4x^5 + 4x^4)\, dx$$

$$= \pi\left[\frac{1}{7}x^7 - \frac{2}{3}x^6 + \frac{4}{5}x^5\right]_0^2$$

$$= \pi\left(\frac{2^7}{7} - \frac{2^7}{3} + \frac{2^7}{5}\right) - 0 = \pi 2^7\left(\frac{1}{7} - \frac{1}{3} + \frac{1}{5}\right) = \frac{128}{105}\pi$$

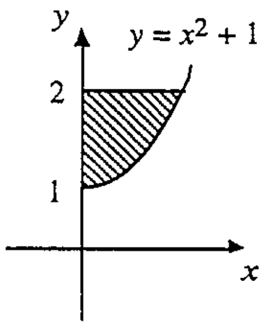
The volume of revolution can also be formed when the area bounded by a curve $y = f(x)$, the y – axis, y = a and y = b is rotated completely about the y – axis.



$$V = \int_a^b \pi x^2 dv$$

Example

Find the volume generated, by rotating about the y - axis, the area enclosed by the curve $y = x^2 + 1$, $x > 0$, the x – axis and the line $y = 2$.



From $y = x^2 + 1$, $x^2 = y - 1$

$$V = \int_1^2 \pi(y - 1)dy \quad = \pi\left[\frac{1}{2}y^2 - y\right]_1^2$$

$$= \pi[(2 - 2) - (1/2 - 1)]$$

$$= \frac{\pi}{2}$$

Exercise 10

1. Find the volumes of solids of revolution formed when the regions bounded by the following curves and the x – axis are rotated through one revolution about the x – axis.
   Sketch the curves.

   (a) $y = \frac{4}{x}$, $x = 1$ and $x = 4$ (b) $y = \sqrt{x}$, $x = 0$ and $x = 4$

   (c) $x + 2y = 2$, $x = 0$ and $x = 2$ (d) $y = \frac{1}{x^2}$, $x = \frac{1}{3}$ and $x = \frac{1}{2}$

   (e) $y = x(x - 1)$ (f) $y = \sqrt{(9 - x^2)}$

   (g) $y^2 = 8x$, $x = 0$ and $x = 4$ (h) $y = \sin x$, $x = 0$ and $x = \pi$

2. Find the volumes of solids of revolution formed when the regions in the first quadrant bounded by the following curves and the y – axis are rotated through one revolution about the y - axis.
   Sketch the curves.

   (a) $x = \sqrt{y}$ and $y = 4$ (b) $x = y^2$ and $y = 1$

   (c) $xy = 1$, $y = 3$ and $y = 6$ (d) $y = 4 - x^2$, $y = -4$ and $y = 4$

   (e) $xy^2 = 2$, $y = 2$ and $y = 4$ (f) $x = y^2 + 1$, $y = -1$ and $y = 1$

   (g) $y = \ln x$, $y = 2$ and $y = 5$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 181 Exercise 4.1:4 Questions 16(Area only).

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 304 Exercise 12B Question 15.

Answers

Exercise 1 Page 29

1. $2x^4 + c$
2. $-\frac{3}{x^2} + c$
3. $\frac{2}{3}x^{\frac{3}{2}} + c$ (etc.)
4. $2\sqrt{x}$
5. $x^3 - \frac{6}{x}$
6. $2x^{\frac{1}{2}} - 2x^{\frac{3}{2}}$
7. $\frac{(3x+4)^5}{15}$
8. $-\frac{(1-2x)^5}{10}$
9. $-\frac{1}{2(2x-3)}$
10. $\frac{\sqrt{4x+1}}{2}$
11. $-\frac{2}{\sqrt{3x+1}}$
12. $-\frac{1}{3}\cos 3x$
13. $3\sin\frac{1}{3}x$
14. $\frac{1}{2}x - \frac{1}{4}\sin 2x$
15. $\frac{1}{2}x + \frac{1}{2}\sin x$

Exercise 2 Page 31

1. $\frac{1}{5}e^{5x}$
2. $-\frac{1}{2}e^{-2x}$
3. $6e^{\frac{1}{2}x}$
4. $\frac{1}{3}\ln 3x$
5. $\ln(x+5)$
6. $\frac{1}{2}\ln(2x-3)$
7. $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}$
8. $x - e^{-x}$
9. $\frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x}$
10. $2\ln(3x+2)$
11. $-\frac{3}{2}\ln(1-2x)$
12. $-\frac{5}{7}\ln(6-7x)$
13. $\frac{1}{4}\tan 4x$
14. $\frac{1}{2}\tan(\pi + 2x)$
15. $\frac{3}{2}\tan 2x$

Exercise 3 Page 33

1. $-\frac{11}{24}$
2. $-\frac{5}{12}$
3. $\frac{154}{5}$
4. $\frac{38}{3}$
5. $\frac{1}{36}$
6. $11\frac{1}{4}$
7. $\frac{1}{2}$
8. 1
9. π
10. $\frac{1}{3}\left(1-\frac{1}{e^6}\right)$
11. $e - 1$
12. $2e^{\frac{1}{2}} - 2$
13. $\ln 3$
14. $\frac{1}{3}\ln\frac{5}{2}$
15. $\ln 3$

Exercise 4 Page 36

1. $\frac{1}{12}(x^2-3)^6$
2. $\frac{1}{9}(x^3-1)^3$
3. $-\frac{1}{3}(1-x^2)^{\frac{3}{2}}$
4. $\frac{1}{5}\sin^5 x$
5. $\frac{1}{4(1-x^2)^2}$
6. $\frac{1}{4}\ln(1+x^4)$
7. $\frac{1}{2}\sec^2 x$
8. $-\frac{1}{5\sin^5 x}$
9. $\frac{1}{3}\ln(3e^x - 1)$
10. $\ln(\tan x)$

Exercise 5 Page 37

1. $\frac{1}{5}(3x+2)^5 - \frac{1}{2}(3x+2)^4 + c$
2. $\frac{1}{4}(2x+3)^7 - \frac{7}{8}(2x+3)^6 + c$
3. $(1+x)^{1/2} + c$
4. $\frac{1}{2}(2x+3)^{3/2} - \frac{9}{2}(2x+3)^{1/2} + c$

Exercise 6 Page 38

1. $-\sqrt{(1-x^2)} + c$
2. $2\sin^{-1}(x/2) + x\frac{\sqrt{(4-x^2)}}{2} + c$
3. $-\sqrt{(9-x^2)} + c$
4. $2\sin^{-1}(x/2) - \frac{x}{2}\sqrt{(4-x^2)} + c$

Exercise 7 Page 40

1. (a) 12 (b) $2\frac{1}{4}$
2. (a) $20\frac{5}{6}$ (b) $3\frac{1}{12}$

Exercise 8 Page 41

1. 36
2. $2\frac{2}{3}$
3. $5\frac{1}{3}$
4. 8
5. $2\sqrt{2}$
6. $\frac{16a^2}{3}$

Exercise 9 Page 42

1. 9
2. $11\frac{1}{4}$
3. $2\sqrt{3} - 2\sqrt{2}$
4. $\frac{2}{3}$
5. $7\frac{1}{2}$
6. $e^2(e^3 - 1)$

Exercise 10 Page 44

1. (a) 12π (b) 8π (c) $\frac{2\pi}{3}$ (d) $\frac{19\pi}{3}$
   (e) $\frac{1\pi}{30}$ (f) 36π (g) 64π (h) $\frac{1\pi^2}{2}$
2. (a) 8π (b) $\frac{1\pi}{5}$ (c) $\frac{1\pi}{6}$ (d) 32π
   (e) $\frac{7\pi}{48}$ (f) $\frac{56\pi}{15}$ (g) $\frac{1}{2}(e^{10} - e^4)\pi$

Use further integration techniques

(a) **Integrals of the form** $\int \frac{1}{\sqrt{1-x^2}}\,dx$ **and** $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

(i) If $f(x) = \sin^{-1} x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$

then $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$

(ii) If $f(x) = \sin^{-1}\left(\frac{x}{2}\right)$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{\left(1-\left(\frac{x}{2}\right)^2\right)}} \times \frac{1}{2} = \frac{1}{\sqrt{\left(1-\frac{x^2}{4}\right)}} \times \frac{1}{2} = \frac{1}{\sqrt{\left(\frac{4-x^2}{4}\right)}} \times \frac{1}{2}$$

$$= \frac{1}{\frac{\sqrt{4-x^2}}{2}} \times \frac{1}{2} = \frac{1}{\sqrt{4-x^2}}$$

then $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + c$

Alternatively, integrate $\int \frac{1}{\sqrt{4-x^2}}\,dx$ by substitution.

Let $x = 2t$, $dx = 2dt$ and $4 - x^2 = 4 - 4t^2 = 4(1 - t^2)$

and $\sqrt{(4 - x^2)} = \sqrt{4(1 - t^2)} = 2\sqrt{(1 - t^2)}$

The integral $\int \frac{1}{\sqrt{4-x^2}}\,dx$ becomes $\int \frac{1}{2\sqrt{1-t^2}}\,2dt$

$$= \int \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \sin^{-1} t + c$$

But $t = \left(\frac{x}{2}\right)$ and so $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + c$

In general

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c}$$

(b) **Integrals of the form** $\int \frac{1}{1+x^2}dx$ **and** $\int \frac{1}{a^2+x^2}dx$

(i) If $f(x) = \tan^{-1}x \Rightarrow f'(x) = \frac{1}{1+x^2}$

then $\int \frac{1}{1+x^2}dx = \tan^{-1}x + c$

(ii) If $f(x) = \tan^{-1}\left(\frac{x}{3}\right)$

and $f'(x) = \frac{1}{\left(1+\left(\frac{x}{3}\right)^2\right)} \times \frac{1}{3} = \frac{1}{\left(1+\frac{x^2}{9}\right)} \times \frac{1}{3} = \frac{1}{\left(\frac{9+x^2}{9}\right)} \times \frac{1}{3}$

$= \frac{3}{9+x^2}$

then $\int \frac{3}{9+x^2}dx = 3\int \frac{1}{9+x^2}dx = \tan^{-1}\left(\frac{x}{3}\right) + c$

so $\int \frac{1}{9+x^2}dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

Alternatively, integrate $\int \frac{1}{9+x^2}dx$ by substitution.

Let $x = 3t$, $dx = 3dt$ and $9 + x^2 = 9 + 9t^2 = 9(1 + t^2)$

The integral $\int \frac{1}{9+x^2}dx$ becomes $\int \frac{1}{9(1+t^2)}3dt$

$= \frac{1}{3}\int \frac{1}{1+t^2}dt = \tan^{-1}t + c$

But $t = \frac{x}{3}$ and so $\int \frac{1}{9+x^2}dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

In general $$\boxed{\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c}$$

Examples Integrate the following :-

(1) $\int \frac{1}{\sqrt{16-x^2}}dx = \sin^{-1}\left(\frac{x}{4}\right) + c$

(2) $\int \frac{1}{16+x^2}dx = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$

(3) $$\int \frac{1}{\sqrt{25-9x^2}}\,dx = \int \frac{1}{\sqrt{9\left(\frac{25}{9}-x^2\right)}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{\frac{25}{9}-x^2}}\,dx$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{x}{\frac{5}{3}}\right) + c$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{5}\right) + c$$

(4) $$\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{4\left(\frac{9}{4}+x^2\right)}\,dx = \frac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx$$
$$= \frac{1}{4}\times\frac{1}{\frac{3}{2}}\tan^{-1}\left(\frac{x}{\frac{3}{2}}\right) + c$$
$$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c$$

Exercise 1

Integrate

1. $\int \frac{1}{\sqrt{49-x^2}}\,dx$
2. $\int \frac{1}{49+x^2}\,dx$
3. $\int \frac{1}{\sqrt{9-x^2}}\,dx$
4. $\int \frac{1}{100+x^2}\,dx$
5. $\int \frac{1}{\sqrt{36-25x^2}}\,dx$
6. $\int \frac{1}{36+25x^2}\,dx$
7. $\int \frac{2}{25+4x^2}\,dx$
8. $\int \frac{3}{\sqrt{36-9x^2}}\,dx$

Evaluate

9. $\int_1^{\sqrt{3}} \frac{2}{1+x^2}\,dx$
10. $\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx$
11. $\int_{\frac{1}{2}}^{1} \frac{3}{\sqrt{1-x^2}}\,dx$
12. $\int_0^{3} \frac{1}{9+x^2}\,dx$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

Page 193 Exercise 4.2:3 Questions 16, 17, 19, 20.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 376 Exercise 15H Questions 13 – 24.

(c) **Integrals by means of Partial Fractions.**

We already know how to express a rational function as a sum of partial fractions from Unit 1, Outcome 1.

We will apply these techniques to integrating rational functions.

__Examples__ (where the denominator has linear and different factors).

(1) $$\int \frac{x+16}{2x^2+x-6}\,dx$$

Let $\dfrac{x+16}{2x^2+x-6} = \dfrac{x+16}{(2x-3)(x+2)} = \dfrac{A}{2x-3} + \dfrac{B}{x+2}$

Multiply both sides by $(2x-3)(x+2)$ => $x + 16 = A(x+2) + B(2x-3)$

Replace with $x = \frac{3}{2}$: $\frac{35}{2} = \frac{7}{2}A$ => $A = 5$

Replace with $x = -2$: $14 = -7B$ => $B = -2$

The integral becomes:

$$\int \frac{x+16}{2x^2+x-6}\,dx = \int\left(\frac{5}{2x-3} - \frac{2}{x+2}\right)dx$$

$$= 5\int \frac{1}{2x-3}\,dx - 2\int \frac{1}{x+2}\,dx$$

$$= \frac{5}{2}\ln(2x-3) - 2\ln(x+2) + c$$

(2) $$\int \frac{2x^2+4x}{(x-1)(x+1)(2x+1)}\,dx$$

Let $\dfrac{2x^2+4x}{(x-1)(x+1)(2x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{2x+1}$

Multiply both sides by $(x-1)(x+1)(2x-1)$

=> $2x^2 + 4x = A(x+1)(2x+1) + B(x-1)(2x+1) + C(x-1)(x+1)$

Put $x = 1$: $6 = 6A$ => $A = 1$

Put $x = -1$: $-2 = 2B$ => $B = -1$

Put $x = -\frac{1}{2}$: $-\frac{3}{2} = -\frac{3}{4}C$ => $C = 2$

The integral becomes:–

$$\int \frac{2x^2+4x}{(x-1)(x+1)(2x+1)}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1} + \frac{2}{2x+1}\,dx$$

contd...

$$\int \frac{1}{x-1} - \frac{1}{x+1} + \frac{2}{2x+1}\,dx = \ln(x-1) - \ln(x+1) + \ln(2x+1) + c$$

Note: You may find the following method of simplifying the right -hand side useful when you come to first order differential equations in a later outcome.

When all integrals are logarithms, c can be replaced by $\ln K$ since they are both constants in different forms.

$$\begin{aligned} \text{and so} \quad & \ln(x-1) - \ln(x+1) + \ln(2x+1) + c \\ = \quad & \ln(x-1) - \ln(x+1) + \ln(2x+1) + \ln K \\ = \quad & \ln\left(\frac{K(x-1)(2x+1)}{(x+1)}\right) \text{ by the laws of logarithms} \end{aligned}$$

(3) $\displaystyle\int \frac{2x^3+7x^2-2x-2}{2x^2+x-6}\,dx$

Use long division

$$\begin{array}{r} x+3 \\ 2x^2+x-6\,\big)\overline{2x^3+7x^2-2x-2} \\ \underline{2x^3+x^2-6x} \\ 6x^2+4x-2 \\ \underline{6x^2+3x-18} \\ x+16 \end{array}$$

The integral can now be written as

$$\int \frac{2x^3+7x^2-2x-2}{2x^2+x-6}\,dx$$

$$= \int\left(x+3+\frac{x+16}{2x^2+x-6}\right)dx = \int\left(x+3+\frac{x+16}{(2x-3)(x+2)}\right)dx$$

Now express $\dfrac{x+16}{(2x-3)(x+2)}$ in its partial fractions:-

Let $\dfrac{x+16}{(2x-3)(x+2)} = \dfrac{A}{2x-3} + \dfrac{B}{x+2}$

Multiply both sides by $(2x-3)(x+2)$

$\Rightarrow \quad x+16 = A(x+2) + B(2x-3)$

Put $x = \frac{3}{2}$: $\quad \frac{35}{2} = \frac{7}{2}A \quad \Rightarrow A = 5$

Put $x = -2$: $\quad 14 = -7B \quad \Rightarrow B = -2$

The integral becomes:

$$\int \frac{2x^3+7x^2-2x-2}{2x^2+x-6}\,dx = \int\left(x+3+\frac{5}{2x-3}-\frac{2}{x+2}\right)dx$$

$$= \frac{1}{2}x^2 + 3x + \frac{5}{2}\ln(2x-3) - 2\ln(x+2) + c$$

Exercise 2

Integrate

1. $\int \frac{x+8}{(x+2)(x+4)}\,dx$

2. $\int \frac{x^2}{x^2-4}\,dx$

3. $\int \frac{x^2-6x-7}{(x-1)(x-2)(x+3)}\,dx$

4. $\int \frac{x^3-2x-13}{x^2-2x-3}\,dx$

Example where the denominator has linear and repeated factors.

(1) $\int \frac{-8x^2+14x-15}{(2x-1)^2(x+2)}\,dx$

Let $\frac{-8x^2+14x-15}{(2x-1)^2(x+2)} = \frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{C}{x+2}$

Multiply both sides by $(2x-1)^2(x+2)$

=> $-8x^2+14x-15 = A(2x-1)(x+2)+B(x+2)+C(2x-1)^2$

Put $x=\frac{1}{2}$: $-10 = \frac{5}{2}B$ => $B=-4$

Put $x=-2$: $-75 = 25C$ => $C=-3$

Put $x=0$: $-15 = -2A+2B+C$

$-15 = -2A-8-3$ => $A=2$

The integral becomes:

$$\int \frac{-8x^2+14x-15}{(2x-1)^2(x+2)}\,dx = \int\left(\frac{2}{2x-1}-\frac{4}{(2x-1)^2}-\frac{3}{x+2}\right)dx$$

$$= \ln(2x-1)+\frac{2}{2x-1}-3\ln(x+2) + c$$

Exercise 3

Integrate

1. $\int \frac{3x^2+x+1}{x(x+1)^2}\,dx$

2. $\int \frac{x^2-2x+10}{(x+2)(x-1)^2}\,dx$

3. $\int \frac{25}{(2x-1)^2(x+2)}\,dx$

4. $\int \frac{5x+2}{(x-2)^2(x+1)}\,dx$

Examples where the denominator has a linear factor and an irreducible quadratic factor of the form $x^2 + a^2$.

(1) $$\int \frac{x-1}{(x+1)(x^2+1)}\,dx$$

Let $\dfrac{x-1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

Multiply both sides by $(x+1)(x^2+1)$

$\Rightarrow \quad x - 1 = A(x^2+1) + (x+1)(Bx+C)$

Put $x = -1$: $\quad -2 = 2A \quad \Rightarrow \quad A = -1$

Put $x = 0$: $\quad -1 = A + C \quad \Rightarrow \quad C = 0$

Put $x = 1$: $\quad 0 = 2A + 2(B+C) \quad \Rightarrow \quad B = 1$

The integral becomes:

$$\int \frac{x-1}{(x+1)(x^2+1)}\,dx = \int\left(\frac{-1}{x+1} + \frac{x}{x^2+1}\right)dx$$

$$= -\ln(x+1) + \frac{1}{2}\ln(x^2+1) + c$$

(2) $$\int \frac{x^2+2x+1}{x(x^2+1)}\,dx$$

Let $\dfrac{x^2+2x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

Multiply both sides by $x(x^2+1)$

$x^2 + 2x + 1 = A(x^2+1) + x(Bx+C)$

Put $x = 0$: $\quad 1 = A \quad \Rightarrow \quad A = 1$

Put $x = 1$: $\quad 4 = 2A + B + C \quad \Rightarrow B + C = 2$

Put $x = -1$: $\quad 0 = 2A + B - C \quad \Rightarrow B - C = -2$

$\left.\begin{matrix}B + C = 2\\ B - C = -2\end{matrix}\right\}$ Solve giving $B = 0,\ C = 2$

The integral becomes:

$$\int \frac{x^2+2x+1}{x(x^2+1)}\,dx = \int\left(\frac{1}{x} + \frac{2}{x^2+1}\right)dx$$

$$= \ln x + 2\tan^{-1}x + c$$

(3) $\int \frac{1}{(x-2)(x^2+1)}\,dx$

Let $\frac{1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$

=> Multiply both sides by $(x-2)(x^2+1)$

$$1 = A(x^2+1) + (x-2)(Bx+C)$$

Put $x = 2$: $1 = 5A$ => $A = \frac{1}{5}$

Put $x = 0$: $1 = A - 2C$ => $C = \frac{-2}{5}$

Put $x = 1$: $1 = 2A - B - C$ => $B = \frac{-1}{5}$

The integral becomes:

$$\int \frac{1}{(x-2)(x^2+1)}\,dx = \int \left(\frac{\frac{1}{5}}{x-2} + \frac{-\frac{1}{5}x - \frac{2}{5}}{x^2+1} \right) dx$$

$$= \int \frac{1}{5(x-2)} - \frac{1}{5}\left(\frac{x+2}{x^2+1}\right) dx$$

$$= \int \frac{1}{5(x-2)} - \frac{1}{5}\left(\frac{x}{x^2+1} + \frac{2}{x^2+1}\right) dx$$

$$= \int \frac{1}{5(x-2)} - \frac{1}{5}\left(\frac{x}{x^2+1}\right) - \frac{1}{5}\left(\frac{2}{x^2+1}\right) dx$$

$$= \int \frac{1}{5(x-2)} - \frac{1}{10}\left(\frac{2x}{x^2+1}\right) - \frac{2}{5}\left(\frac{1}{x^2+1}\right) dx$$

$$= \frac{1}{5}\ln(x-2) - \frac{1}{10}\ln(x^2+1) - \frac{2}{5}\tan^{-1}x + c$$

Exercise 4

Integrate :-

1. $\int \frac{3x+1}{(x-1)(x^2+1)}\,dx$

2. $\int \frac{3x^2+92x}{(x+6)(x^2+1)}\,dx$

3. $\int \frac{x}{x^4-1}\,dx$

4. $\int \frac{x}{(x+1)(x^2+4)}\,dx$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

Page 196 Exercise 4.2:5 Questions 7, 8(i),(ii),(iii) only, 9, 10.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 488 Exercise 19B Questions 46, 47, 48.
Page 494 Exercise 19D Question 6.

Integration by Parts

We have already used a formula to differentiate a product of two functions $u(x)$ and $v(x)$.

i.e. $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides of the above formula with respect to x

=> $$\int\left(\frac{d}{dx}(uv)\right)dx = \int\left(u\frac{dv}{dx}\right)dx + \int\left(v\frac{du}{dx}\right)dx$$

i.e. $$uv = \int\left(u\frac{dv}{dx}\right)dx + \int\left(v\frac{du}{dx}\right)dx$$

Rearranging gives

$$\boxed{\int\left(u\frac{dv}{dx}\right)dx = uv - \int\left(v\frac{du}{dx}\right)dx}$$ or simply $$\int u\,dv = uv - \int v\,du$$

This formula enables us to integrate a product of two functions and the method is known as **integration by parts**.

N.B. When choosing which function is u and which is $\frac{dv}{dx}$, care must be taken to ensure that $v\frac{du}{dx}$ is simple enough to integrate.

Examples

(1) $\int x\cos x\,dx$

Let $u = x$ and $dv = \cos x\,dx$

$du = 1dx$ and $v = \sin x$

$$\therefore \int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + c$$

Note if we try to let $u = \cos x$ and $dv = x\,dx$ instead, this will lead to :-

$du = -\sin x\,dx$ and $v = \frac{1}{2}x^2$

$$\therefore \int x\cos x\,dx = \frac{1}{2}x^2\cos x + \int \frac{1}{2}x^2 \sin x\,dx$$

This time, the second integral is not any simpler than the original.

(2) $\int xe^x dx$

Let $u = x$ and $dv = e^x dx$

$du = 1dx$ and $v = e^x$

$$\therefore \int xe^x dx = xe^x - \int e^x dx$$
$$= xe^x - e^x + c$$

(3) $\int x\ln x\,dx$

Let $u = \ln x$ and $dv = x\,dx$

$du = \frac{1}{x}dx$ and $v = \frac{1}{2}x^2$

$$\therefore \int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{x}\times\frac{1}{2}x^2 dx$$
$$= \frac{1}{2}x^2\ln x - \int \frac{x}{2}dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$$

Exercise 5

Integrate :-

1. $\int x\sin x dx$ 2. $\int x\sin 3x dx$

3. $\int \sqrt{x}\ln x dx$ 4. $\int \frac{1}{x^3}\ln x dx$

5. $\int xe^x dx$ 6. $\int x\cos 4x dx$

Integration by Parts involving Repeated Applications.

Examples

(1) $\int x^2\sin x dx$ Let $u = x^2$ and $dv = \sin x dx$

$du = 2x dx$ and $v = -\cos x$

$$\therefore \int x^2 \sin x dx = -x^2\cos x + \int 2x\cos x dx$$

Let $u = 2x$ and $dv = \cos x dx$

$du = 2dx$ and $v = \sin x$

$$= -x^2\cos x + 2x\sin x - \int 2\sin x dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

(2) $\int x^2 e^x dx$ Let $u = x^2$ and $dv = e^x dx$

$du = 2x dx$ and $v = e^x$

$$\therefore \int x^2 e^x dx = x^2e^x - \int 2xe^x dx$$

Let $u = 2x$ and $dv = e^x dx$

$du = 2dx$ and $v = e^x$

$$= x^2e^x - \left[2xe^x - \int 2e^x dx\right]$$

$$= x^2e^x - 2xe^x + 2e^x + c$$

Exercise 6

Integrate :-

1. $\int x^2\cos x dx$ 2. $\int x^2\sin 3x dx$

3. $\int x^2e^{2x} dx$ 4. $\int x^2\cos 2x dx$

5. $\int x^2e^{-x} dx$ 6. $\int x^3e^x dx$

Integration by Parts involving a "Dummy" Function.

Functions like $\ln x$, $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ do not have a standard integral but have a standard derivative.

In order to integrate them, we introduce a "dummy" function, namely the number 1 (one).

i.e. for $u = \ln x$, $u = \sin^{-1}x$ or $u = \tan^{-1}x$, let $dv = 1$ each time.

Examples

(1) $\int \ln x\,dx$

Let $u = \ln x$ and $dv = 1dx$

$du = \frac{1}{x}dx$ and $v = x$

$$\therefore \int \ln x\,dx = x\ln x - \int x \times \frac{1}{x}dx$$
$$= x\ln x - \int 1dx$$
$$= x\ln x - x + c$$

(2) $\int \sin^{-1}dx$

Let $u = \sin^{-1}x$ and $dv = 1dx$

$du = \frac{1}{\sqrt{1-x^2}}dx$ and $v = x$

$$\therefore \int \sin^{-1}dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$$

For $\int \frac{x}{\sqrt{1-x^2}}dx$, use the substitution $t = 1 - x^2 \Rightarrow dt = -2xdx$

i.e. $xdx = -\frac{1}{2}dt$

The integral becomes:

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int -\frac{1}{2\sqrt{t}}dt = -\frac{1}{2}\int t^{-\frac{1}{2}}dt = -\frac{1}{2}2t^{\frac{1}{2}} + c$$
$$= -\sqrt{1-x^2} + c$$

$$\therefore \int \sin^{-1}dx = x\sin^{-1}x + \sqrt{1-x^2} + c$$

Exercise 7

Integrate :-

1. $\int \tan^{-1}xdx$
2. $\int \sin^{-1}3xdx$
3. $\int \tan^{-1}2xdx$
4. $\int \sin^{-1}\frac{1}{2}xdx$
5. $\int \cos^{-1}xdx$
6. $\int \tan^{-1}\frac{1}{2}xdx$
7. $\int \ln 2xdx$
8. $\int (\ln x)^2dx$

Integration by Parts involving an Integral that returns to its original form.

There are some integrals which return to themselves when integration by parts is used.
We have a special way of dealing with them.

Examples

(1) $\int e^x \sin x dx$

Let $u = e^x$ and $dv = \sin x dx$
$du = e^x dx$ and $v = -\cos x$

$$\therefore \quad \int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$$

Let $u = e^x$ and $dv = \cos x$
$du = e^x$ and $v = \sin x$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

which is the original integral

$$\therefore \quad \int e^x \sin x dx + \int e^x \sin x dx = e^x \sin x - e^x \cos x$$

i.e. $\Rightarrow \quad 2\int e^x \sin x dx = e^x \sin x - e^x \cos x$

$$\Rightarrow \quad \int e^x \sin x dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + c$$

(2) $\int e^{2x} \cos x dx$

Let $u = e^{2x}$ and $dv = \cos x\, dx$
$du = 2e^{2x} dx$ and $v = \sin x$

$$\therefore \quad \int e^{2x} \cos x dx = e^{2x} \sin x - 2\int e^{2x} \sin x dx$$

Let $u = e^{2x}$ and $dv = \sin x\, dx$
$du = 2e^{2x}\, dx$ and $v = -\cos x$

$$= e^{2x} \sin x - 2[-e^{2x} \cos x + 2\int e^{2x} \cos x dx]$$

$$= e^{2x} \sin x + 2e^{2x} \cos x - 4\int e^{2x} \cos x dx$$

which is the original integral

i.e. $\Rightarrow \quad 5\int e^{2x} \cos x dx = e^{2x} \sin x + 2e^{2x} \cos x$

$$\Rightarrow \quad \int e^{2x} \cos x dx = \frac{1}{5}(e^{2x} \sin x + 2e^{2x} \cos x) + c$$

Exercise 8

Integrate

1. $\int e^x \sin 2x dx$
2. $\int e^{-x} \sin x dx$
3. $\int e^{-2x} \cos 3x dx$
4. $\int e^x \cos^2 x dx$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 199 Exercise 4.2:6 Questions 1, 2, 4, 5, 8, 12 .
Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 505 Exercise 20B Questions 1 – 25.

Answers

Exercise 1

1. $\sin^{-1}(\frac{x}{7}) + c$
2. $\frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right) + c$
3. $\sin^{-1}\left(\frac{x}{3}\right) + c$
4. $\frac{1}{10}\tan^{-1}\left(\frac{x}{10}\right) + c$
5. $\frac{1}{5}\sin^{-1}\left(\frac{5x}{6}\right) + c$
6. $\frac{1}{30}\tan^{-1}\left(\frac{5x}{6}\right) + c$
7. $\frac{1}{5}\tan^{-1}\left(\frac{2x}{5}\right) + c$
8. $\sin^{-1}\left(\frac{x}{2}\right) + c$
9. $\frac{\pi}{6}$
10. $\frac{\pi}{4}$
11. π
12. $\frac{\pi}{12}$

Exercise 2

1. $3\ln(x+2) - 2\ln(x+4) + c$
2. $x - \ln(x+2) + \ln(x-2) + c$
3. $3\ln(x-1) - 3\ln(x-2) + \ln(x+3) + c$
4. $\frac{1}{2}x^2 + 2x + 2\ln(x-3) + 3\ln(x+1) + c$

Exercise 3

1. $\ln x + 2\ln(x+1) + \frac{3}{x+1} + c$
2. $2\ln(x+2) - \ln(x-1) - \frac{3}{x-1} + c$
3. $\ln(x+2) - \ln(2x-1) - \frac{5}{2x-1} + c$
4. $\frac{1}{3}\ln(x-2) - \frac{1}{3}\ln(x+1) - \frac{4}{x-2} + c$

Exercise 4

1. $2\ln(x-1) - \ln(x^2+1) + \tan^{-1}x + c$
2. $\frac{15}{2}\ln(x^2+1) - 12\ln(x+6) + 2\tan^{-1}x + c$
3. $\frac{1}{4}\ln(x+1) + \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x^2+1) + c$
4. $\frac{1}{10}\ln(x^2+4) - \frac{1}{5}\ln(x+1) + \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + c$

Exercise 5

1. $-x\cos x + \sin x + c$
2. $-\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x + c$
3. $\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + c$
4. $-\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + c$
5. $xe^x - e^x + c$
6. $\frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + c$

Exercise 6

1. $x^2\sin x + 2x\cos x - 2\sin x + c$
2. $-\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + c$
3. $\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$
4. $\frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + c$
5. $-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$
6. $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c$

Exercise 7

1. $x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$
2. $x\sin^{-1}3x + \frac{1}{3}\sqrt{(1-9x^2)} + c$
3. $x\tan^{-1}2x - \frac{1}{4}\ln(1+4x^2) + c$
4. $x\sin^{-1}\frac{1}{2}x + \sqrt{(4-x^2)} + c$
5. $x\cos^{-1}x - \sqrt{(1-x^2)} + c$
6. $x\tan^{-1}\frac{1}{2}x - \ln(4+x^2) + c$
7. $x\ln 2x - x + c$
8. $x(\ln x)^2 - 2x\ln x + 2x + c$

Exercise 8

1. $\frac{1}{5}e^x\sin 2x - \frac{2}{5}e^x\cos 2x + c$
2. $-\frac{1}{2}e^{-x}\cos x - \frac{1}{2}e^{-x}\sin x + c$
3. $\frac{3}{13}e^{-2x}\sin 3x - \frac{2}{13}e^{-2x}\cos 3x + c$
4. $e^x\cos^2 x + \frac{1}{5}e^x\sin 2x - \frac{2}{5}e^x\cos 2x + c$

or $\frac{1}{2}e^x + \frac{1}{10}e^x\cos 2x + \frac{1}{5}e^x\sin 2x + c$

Exercise 5A (page 77)

1 $\frac{665}{24}$

2 $\frac{1}{2}\ln\left(\frac{92}{17}\right)$

3 $8\ln 2 - 4$

4 $1 - \frac{\sqrt{3}}{2}$

5 $\frac{1}{6}(5\sqrt{5} - 3\sqrt{3})$

6 $\ln\left(\frac{3}{2}\right)$

7 $\ln\left(\frac{\pi}{4}\right)$

8 $\ln\left(\frac{2}{\sqrt{3}}\right)$

9 $\frac{31}{160}$

10 $\sqrt{2} - \frac{2}{\sqrt{3}}$

11 $\ln 2$

12 $\frac{e^2}{2(e^4 - 1)}$

Exercise 5B (page 77)

1 $\frac{8}{15}$

2 $-\frac{7}{60\sqrt{2}} + \frac{2}{15}$

3 a $\frac{8}{u^2-4} = \frac{8}{(u+2)(u-2)}$ b $2 + 2\ln\left(\frac{5}{3}\right)$

Advanced Higher Maths

Integration

2001

(a) Obtain partial fractions for

$$\frac{x}{x^2-1},\ x>1.$$

(b) Use the result of (a) to find

$$\int \frac{x^3}{x^2-1}\,dx,\ x>1.$$

(2, 4 marks)

2002

Use the substitution $x+2=2\tan\theta$ to obtain $\int \frac{1}{x^2+4x+8}\,dx$. **(5 marks)**

2003

Use the substitution $x=1+\sin\theta$ to evaluate $\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)^3}\,d\theta$. **(5 marks)**

2004

① Express $\frac{1}{x^2-x-6}$ in partial fractions.

Evaluate $\int_0^1 \frac{1}{x^2-x-6}\,dx$. **(2, 4 marks)**

② A solid is formed by rotating the curve $y=e^{-2x}$ between $x=0$ and $x=1$ through 360° about the x-axis. Calculate the volume of the solid that is formed. **(5 marks)**

2005

① Use the substitution $u=1+x$ to evaluate $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$. **(5 marks)**

② Express $\frac{1}{x^3+x}$ in partial fractions.

Obtain a formula for $I(k)$, where $I(k)=\int_1^k \frac{1}{x^3+x}\,dx$, expressing it in the form $\ln\frac{a}{b}$, where a and b depend on k.

Write down an expression for $e^{I(k)}$ and obtain the value of $\lim_{k\to\infty} e^{I(k)}$. **(4, 4, 2 marks)**

③ (a) Given $f(x)=\sqrt{\sin x}$, where $0<x<\pi$, obtain $f'(x)$.

(b) If, in general, $f(x)=\sqrt{g(x)}$, where $g(x)>0$, show that $f'(x)=\dfrac{g'(x)}{k\sqrt{g(x)}}$,

stating the value of k.

Hence, or otherwise, find $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$. **(1, 2, 3 marks)**

2006

Find $\displaystyle\int \frac{12x^3-6x}{x^4-x^2+1}\,dx$. **(3 marks)**

2007

① Express $\dfrac{2x^2-9x-6}{x\left(x^2-x-6\right)}$ in partial fractions.

Given that $\displaystyle\int_4^6 \frac{2x^2-9x-6}{x\left(x^2-x-6\right)}\,dx=\ln\frac{m}{n}$,

determine values for the integers m and n. **(3, 3 marks)**

② Use the substitution $u=1+x^2$ to obtain $\displaystyle\int_0^1 \frac{x^3}{\left(1+x^2\right)^4}\,dx$.

A solid is formed by rotating the curve $y=\dfrac{x^{3/2}}{\left(1+x^2\right)^2}$ between $x=0$ and $x=1$ through

360° about the x-axis. Write down the volume of this solid. **(5, 1 marks)**

2008

① Express $\dfrac{12x^2+20}{x\left(x^2+5\right)}$ in partial fractions.

Hence evaluate $\displaystyle\int_1^2 \frac{12x^2+20}{x\left(x^2+5\right)}\,dx$. **(3, 3 marks)**

② Write down the derivative of $\tan x$.

Show that $1+\tan^2 x=\sec^2 x$.

Hence obtain $\int \tan^2 x\,dx$. **(1, 1, 2 marks)**

2009

① Show that $\displaystyle\int_{\ln 3/2}^{\ln 2} \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx=\ln\frac{9}{5}$. **(4 marks)**

② Use the substitution $x=2\sin\theta$ to obtain the exact value of $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$.

(Note that $\cos 2A=1-2\sin^2 A$.) **(6 marks)**

2010

① Evaluate $$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,dx$$

expressing your answer in the form $\ln\frac{a}{b}$, where a and b are integers. **(6 marks)**

② A new board game has been invented and the symmetrical design on the board is made from 4 identical "petal" shapes. One of these petals is the region enclosed between the curves $y=x^2$ and $y^2=8x$ as shown shaded in diagram 1 below.

Calculate the area of the complete design, as shown in diagram 2.

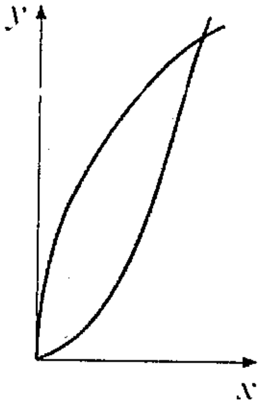


Diagram 1

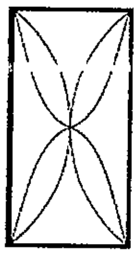


Diagram 2

The counter used in the game is formed by rotating the shaded area shown in diagram 1 above, through $360°$ about the y-axis.
Find the volume of plastic required to make one counter. **(5, 5 marks)**

2011

① Express $\frac{13-x}{x^2+4x-5}$ in partial fractions and hence obtain $\int \frac{13-x}{x^2+4x-5}\,dx$. **(5 marks)**

② Obtain the exact value of $\int_0^{\pi/4}(\sec x - x)(\sec x + x)\,dx$. **(3 marks)**

Advanced Higher Maths

2012

Use the substitution $x = 4\sin\theta$ to evaluate $\int_0^2 \sqrt{16 - x^2}\, dx$. **(6 marks)**

2013

① The velocity, v, of a particle P at time t is given by

$$v = e^{3t} + 2e^t$$

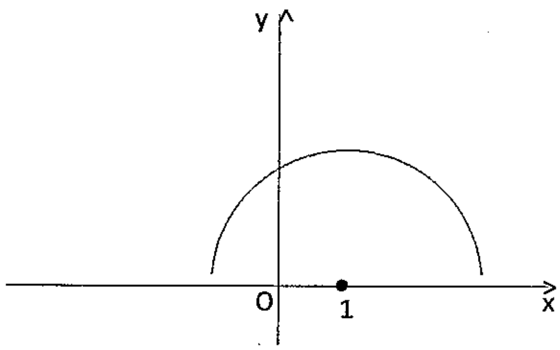
(a) Find the acceleration of P at time t.

(b) Find the distance covered by P between $t = 0$ and $t = \ln 3$. **(2, 3 marks)**

② Integrate $\dfrac{\sec^2 3x}{1 + \tan 3x}$ with respect to x. **(4 marks)**

2014

① A semi-circle with centre $(1, 0)$ and radius 2, lies on the x-axis as shown.
Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis.



(5 marks)

② Use the substitution $x = \tan\theta$ to determine the exact value of

$$\int_0^1 \frac{dx}{\left(1 + x^2\right)^{3/2}}.$$ **(6 marks)**

Advanced Higher Maths

2015

Find $\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}\,dx,\ x > 3.$ **(9 marks)**

Integration

1 Standard Integrals

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad \int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

$$\int \sin x \, dx = -\cos x + c \qquad \int \cos x \, dx = \sin x + c$$

$$\int \sin(ax+b) \, dx = -\tfrac{1}{a}\cos(ax+b) + c \qquad \int \cos(ax+b) \, dx = \tfrac{1}{a}\sin(ax+b) + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c \qquad \int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\tfrac{x}{a}\right) + c$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + c \qquad \int \frac{1}{a^2-x^2} \, dx = \tfrac{1}{a}\tan^{-1}\left(\tfrac{x}{a}\right) + c$$

$$\int e^x \, dx = e^x + c \qquad \int e^{ax+b} \, dx = \tfrac{1}{a}e^{ax+b} + c$$

$$\int \tfrac{1}{x} \, dx = \ln|x| + c \qquad \int \frac{1}{ax+b} \, dx = \tfrac{1}{a}\ln|ax+b| + c$$

2 Integration by Substitution

To find $\int f(x) \, dx$ using the substitution $u = u(x)$

1. find $\dfrac{du}{dx}$ and rearrange to get dx in terms of du;
2. replace dx by this expression and replace $u(x)$ with u;
3. integrate with respect to u.

EXAMPLE

Use the substitution $u = x^2 - 8$ to find $\int x^3 (x^2 - 8)^{\frac{1}{3}}\, dx$.

Given $u = x^2 - 8$, $\frac{du}{dx} = 2x$ so $dx = \frac{du}{2x}$. Then

$$\begin{aligned}
\int x^3 (x^2 - 8)^{\frac{1}{3}}\, dx &= \int x^3 u^{\frac{1}{3}} \frac{du}{2x} \\
&= \tfrac{1}{2} \int x^2 u^{\frac{1}{3}}\, du \\
&= \tfrac{1}{2} \int (u + 8) u^{\frac{1}{3}}\, du \\
&= \tfrac{1}{2} \int \left(u^{\frac{4}{3}} + 8u^{\frac{1}{3}} \right) du \\
&= \tfrac{1}{2} \left(\tfrac{3}{7} u^{\frac{7}{3}} + \tfrac{24}{4} u^{\frac{4}{3}} \right) + c \\
&= \tfrac{3}{14} (x^2 - 8)^{\frac{7}{3}} + 3 (x^2 - 8)^{\frac{4}{3}} + c.
\end{aligned}$$

To find a definite integral, you should change the limits to be in terms of u.

3 Areas

The area between two curves or between a curve and an axis can be calculated as in Higher.

4 Volumes of Solids of Revolution

The solid formed by rotating a curve between $x = a$ and $x = b$ about the x-axis has volume $V = \int_a^b \pi y^2\, dx$.

5 Partial fractions and Integration

Partial fractions can be used to integration rational functions.

EXAMPLE

$$\int \frac{1}{(x+1)(x-3)} dx = \tfrac{1}{4} \int \left(\frac{1}{x-3} - \frac{1}{x+1} \right) dx = \tfrac{1}{4} \left(\ln|x-3| - \ln|x+1| \right) + c.$$