Advanced Higher Maths

Unit 1.3

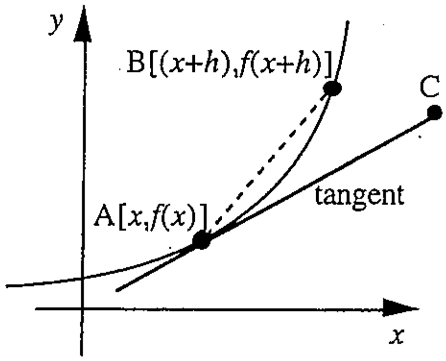
Differentiation 1

Outcome 2 – Differentiation

Differentiate functions involving sums, products, quotients and composites of the elementary functions, x^n $n \in \mathbb{Q}$, $\sin x$, $\cos x$, $\tan x$, e^x and $\ln x$.

Differentiation from First Principles

An approximation for the gradient of the tangent AC to the curve at A can be found by taking a second point B, on the curve and calculating the gradient of the chord AB instead.



If A is the point $[x, f(x)]$ and the other point B is taken as $[x+h, f(x+h)]$ where $|h|$ is small and h can be positive or negative, we can calculate the gradient of the chord AB.

$$m_{AB} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

m_{AB} tends to a **limit as h tends to zero.**
This limit is denoted by $f'(x)$, the derivative of $f(x)$ and gives the gradient of the the tangent AC to the curve $y = f(x)$ at A.

i.e. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. This is known as Differentiation from First Principles.

Example
Find the derivative of x^2 from first principles.
$f(x) = x^2$, $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} = \lim_{h \to 0}(2x + h) = 2x$$

Exercise 1

Differentiate the following functions from First Principles :-

1. (a) $f(x) = x^3$ (b) $f(x) = x^2 + 2x$ (c) $f(x) = 3x^2 + 4x - 5$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 52-53 Exercise 1.3:1 Question 5.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 258 Exercise 10A Question 3.

Standard derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$(ax+b)^n$	$an(ax+b)^{n-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\sin(ax+b)$	$a\cos(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$

All of the above were covered for the Higher course.

Exercise 2

Differentiate the following functions with respect to x :-

1. $f(x) = x^3 - x^2 + 5x - 6$
2. $f(x) = 3x^2 + 7 - \frac{4}{x}$
3. $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$
4. $f(x) = x^{\frac{3}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}}$
5. $f(x) = \frac{1}{x^2} - \frac{1}{x^3}$
6. $f(x) = \frac{\sqrt{x}}{x^2} + \frac{x^2}{\sqrt{x}}$
7. $f(x) = (4x + 5)^5$
8. $f(x) = \left(2x^4 - 3\right)^{\frac{1}{2}}$
9. $f(x) = \frac{3}{\sqrt{\left(4 - x^2\right)}}$
10. $f(x) = \frac{4}{\left(x^3 + 3x\right)^{\frac{1}{3}}}$
11. $f(x) = \cos^3 x$
12. $f(x) = \sqrt{\sin x}$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 54 Exercise 1.3:2 Questions 1,2,3. and Page 160 Exercise 3.4:3 Question 3.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 261 Exercise 10B Questions 1 - 37 and Page 370 Exercise 15F Questions 1-32(only sin/cos)

New Trigonometric functions :-

secantθ(secθ) $= \frac{1}{\cos\theta}$, cosecantθ(cosecθ) $= \frac{1}{\sin\theta}$, cotangentθ(cotθ) $= \frac{1}{\tan\theta}$.

Differentiate a Simple Composite function using the Chain Rule

The Chain rule (An alternative form if not already met)
If y is a function of u, and u is a function of x, then, if y is now regarded as a function of x

=> $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example Differentiate $y = (x^2 + 3x - 5)^5$

Either $\frac{dy}{dx} = 5(x^2 + 3x - 5)^4(2x + 3)$

Or Let $u = x^2 + 3x - 5$ then $y = u^5$ and $u = x^2 + 3x - 5$

$\frac{dy}{du} = 5u^4$ $\frac{du}{dx} = 2x + 3$

then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4(2x + 3) = 5(x^2 + 3x - 5)^4(2x + 3)$

This method could be adopted in more complicated functions, but otherwise use the original method of differentiating the outer function and multiplying by the derivative of the inner function.

Exercise 3

Differentiate the following functions using the Chain Rule as above :-

1. $y = (x^2 + 4x - 5)^3$
2. $y = \sqrt{x^3 + 5}$
3. $y = (1 + 2\sqrt{x})^4$
4. $y = \frac{3}{\sqrt{(4 - x^2)}}$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

Page 64 Exercise 1.3:6 Questions 2.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 319 Exercise 13A Questions 1 - 20.

Differentiate a Product

The Product Rule :- Used to differentiate a product of two functions.
If u and v are functions of x, ie. $u(x)$ and $v(x)$ then :-

$$\frac{d}{dx}[u(x)\times v(x)] = u(x)\times\frac{d}{dx}[v(x)] + v(x)\times\frac{d}{dx}[u(x)]$$

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Examples

1. $y = x\cos x$ Put $u = x$ and $v = \cos x$

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = -x\sin x + \cos x = \cos x - x\sin x$$

2. $y = x^2\sin 3x$ Put $u = x^2$ and $v = \sin 3x$

$\frac{du}{dx} = 2x$ $\frac{dv}{dx} = 3\cos 3x$

$$\frac{dy}{dx} = 3x^2\cos 3x + 2x\sin 3x$$

Exercise 4

Differentiate the following functions using the Product rule as above :-

1. $y = x^2(x-3)^2$
2. $y = x(2x+3)^3$
3. $y = x\sqrt{(x-6)}$
4. $y = \sqrt{x}(x-3)^3$
5. $y = (x+1)^2(x-1)^4$
6. $y = x^3\sqrt{(x-1)}$
7. $y = x\sin x$
8. $y = x^2\sin x$
9. $y = \sin x\cos x$
10. $y = \sin 2x\cos 5x$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 67 Exercise 1.3:7 Questions 1(i), (ii), (iii), (iv), (v) and
Page 160 Exercise 3.4:3 Question 5(i), (ii), (iii).

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 330 Exercise 13E Questions 1 - 13.

Differentiate a quotient

The Quotient Rule :- Used to differentiate a rational function.
If u and v are functions of x, ie. $u(x)$ and $v(x)$ then

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)\frac{d}{dx}[u(x)] - u(x)\frac{d}{dx}[v(x)]}{[v(x)]^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Examples

1. $y = \dfrac{x^2 - 1}{x^2 + 1}$ Put $u = x^2 - 1$ and $v = x^2 + 1$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 2x$$

$$v^2 = (x^2 + 1)^2$$

$$\frac{dy}{dx} = \frac{2x(x^2 + 1) - 2x(x^2 - 1)}{(x^2 + 1)^2} = \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}$$

2. $y = \dfrac{2x}{\sqrt{(x^2 + 1)}}$ Put $u = 2x$ and $v = (x^2 + 1)^{\frac{1}{2}}$

$$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \times 2x$$

$$v^2 = (x^2 + 1)$$

$$\frac{dy}{dx} = \frac{2(x^2 + 1)^{\frac{1}{2}} - 2x^2(x^2 + 1)^{-\frac{1}{2}}}{(x^2 + 1)} = \frac{2(x^2 + 1)^{-\frac{1}{2}}[(x^2 + 1) - x^2]}{(x^2 + 1)} = \frac{2}{(x^2 + 1)^{\frac{3}{2}}}$$

Exercise 5

Differentiate the following functions using the Quotient rule as above :-

1. $y = \dfrac{x^2}{x + 3}$
2. $y = \dfrac{4 - x}{x^2}$
3. $y = \dfrac{4x}{(1 - x)^3}$
4. $y = \dfrac{2x^2}{x - 2}$
5. $y = \dfrac{(1 - 2x)^3}{x^3}$
6. $y = \dfrac{\sqrt{(x + 1)}}{x^2}$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

Page 67 Exercise 1.3:7 Questions 1(vi), (vii), (viii), 8. and
Page 160 Exercise 3.4:3 Questions 5(iv), (v).

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 330 Exercise 13E Questions 14 - 24.

Derivatives of New Functions

The derivatives of $y = \tan x$, $y = \text{cosec}\,x$, $y = \cot x$, $y = e^x$ and $y = \ln x$

1. $y = \tan x = \dfrac{\sin x}{\cos x}$

Put $u = \sin x$ and $v = \cos x$

$\dfrac{du}{dx} = \cos x \qquad \dfrac{dv}{dx} = -\sin x$

$v^2 = \cos^2 x$

$$\frac{dy}{dx} = \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

2. $y = \text{cosec}\,x = \dfrac{1}{\sin x} = (\sin x)^{-1}$

$$\frac{dy}{dx} = -(\sin x)^{-2} \times \cos x = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\text{cosec}\,x\cot x$$

3. $y = \sec x = \dfrac{1}{\cos x} = (\cos x)^{-1}$

$$\frac{dy}{dx} = -(\cos x)^{-2} \times -\sin x = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x\tan x$$

4. $y = \cot x = \dfrac{1}{\tan x} = (\tan x)^{-1}$

$$\frac{dy}{dx} = -(\tan x)^{-2} \times \sec^2 x = -\frac{1}{\tan^2 x} \times \sec^2 x = -\frac{\cos^2 x}{\sin^2 x} \times \frac{1}{\cos^2 x} = -\frac{1}{\sin^2 x} = -\text{cosec}^2 x$$

5. The following result can be obtained.

If $y = e^x$

then $\dfrac{dy}{dx} = e^x$

or $\dfrac{d}{dx}\left(e^x\right) = e^x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h - 1)}{h} \\ &= e^x \lim_{h\to 0}\frac{(e^h - 1)}{h} = e^x \end{aligned}$$

6. It follows that we can now find $\dfrac{d}{dx}(\log_e x)$

If $y = \ln x = \log_e x$ then $x = e^y$ (by definition)

and so $\dfrac{dx}{dy} = e^y$

therefore $\dfrac{dy}{dx} = \dfrac{1}{e^y}$

$\dfrac{dy}{dx} = \dfrac{1}{x}$

or $\dfrac{d}{dx}\left(\log_e x\right) = \dfrac{1}{x}$

Examples

1. $y = \tan 2x$

$\dfrac{dy}{dx} = 2\sec^2 2x$

2. $y = \tan^2 x$

$\dfrac{dy}{dx} = 2\tan x.\sec^2 x$

3. $y = 3\text{cosec}2x$

$\dfrac{dy}{dx} = 3 \times -2\text{cosec}2x\cot 2x$

$= -6\text{cosec}2x\cot 2x$

4. $y = 2\sec 3x$

$\dfrac{dy}{dx} = 2 \times 3\sec 3x\tan 3x$

$= 6\sec 3x\tan 3x$

5. $y = \cot^3 x \quad \dfrac{dy}{dx} = 3\cot^2 x \times -\text{cosec}^2 x = -3\cot^2 x\text{cosec}^2 x$

6. $y = e^{3x}$

$\dfrac{dy}{dx} = 3e^{3x}$ from $y = e^u$ where $u = 3x$

$\dfrac{dy}{dx} = e^u \qquad \dfrac{du}{dx} = 3$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = e^u \times 3 = 3e^{3x}$

In general $\dfrac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$

7. $y = \ln 3x$

$\dfrac{dy}{dx} = 3 \times \dfrac{1}{3x} = \dfrac{1}{x}$ from $y = \ln u$ where $u = 3x$

$\dfrac{dy}{du} = \dfrac{1}{u} \qquad \dfrac{du}{dx} = 3$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = \dfrac{1}{u} \times 3 = \dfrac{1}{3x} \times 3 = \dfrac{1}{x}$

In general $\dfrac{d}{dx}\{ln[f(x)]\} = f'(x) \times \dfrac{1}{f(x)}$

8. $y = e^{x^2}$

$\dfrac{dy}{dx} = 2xe^{x^2}$

9. $y = e^{\sin x}$

$\dfrac{dy}{dx} = \cos x e^{\sin x}$

10. $y = x^2 e^{2x}$ Put $u = x^2$ and $v = e^{2x}$

$\dfrac{du}{dx} = 2x \qquad \dfrac{dv}{dx} = 2e^{2x}$

$\dfrac{dy}{dx} = 2x^2e^{2x} + 2xe^{2x} = 2xe^{2x}(x + 1)$ (The Product Rule)

cont'd .....

11. $y = \frac{\ln x}{x}$ Put $u = \ln x$ and $v = x$

$\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = 1$

$v^2 = x^2$

$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \ln x \times 1}{x^2} = \frac{1 - \ln x}{x^2}$ (The Quotient Rule)

Exercise 6

Differentiate using the Chain, Product and Quotient Rules as above :-

1. $y = \tan^3 2x$
2. $y = -2\text{cosec}^4 x$
3. $y = \sec x \tan x$
4. $y = x^2 \cot x$
5. $y = \ln(3x + 2)$
6. $y = (x + 2)e^{-x}$
7. $y = \frac{e^x}{x + 2}$
8. $y = \frac{x^2}{\ln x}$
9. $y = \ln\left(\sqrt{(x^2 + 1)}\right)$
10. $y = xe^{-2x^2}$
11. $y = \ln\left(\frac{1 + x}{1 - x}\right)$
12. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Page 105 Exercise 2.5:2 Questions 5, 6 and Page 108 Exercise 2.5:4 Questions 8, 10
Page 160 Exercise 3.4:3 Questions 10,11,13 and 16.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)
Page 370 Exercise 15F Questions 1-34(cosec/sec/cot only) and
Page 480 Exercise 19A Questions 1-15 and Page 487 Exercise 19B Questions 1-15, 17-20.

Higher Derivatives

Let $y = x^2 + 3x + 4$

$$\frac{dy}{dx} = 2x + 3$$

Here $\frac{dy}{dx}$ is defined as a function of x and so can be differentiated with respect to x.

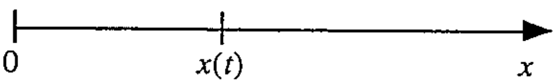
ie. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is usually written as $\frac{d^2y}{dx^2}$, $f''(x)$ or y'' and is called the 2^{nd} **derivative** of y with respect to x.

This process can be repeated to get the 3rd, 4th , ------ , n^{th} derivative which are written as $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, ..., $\frac{d^ny}{dx^n}$

Motion in a straight line

Take the x-axis to be the straight line along which the motion takes place.
The **displacement** is defined as the distance from the origin in time t and is denoted by $x(t)$.



Velocity is defined as the rate of change of displacement with respect to time and is denoted by $v(t)$.

ie. $v(t) = \frac{d}{dt}(x(t))$ or simply $v = \frac{dx}{dt}$

Acceleration is defined as the rate of change of velocity with respect to time and is denoted by $a(t)$.

ie. $a(t) = \frac{d}{dt}(v(t))$ or simply $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

If this has a positive value, it is called acceleration and if negative it is called a deceleration.

Another notation used is

x - displacement

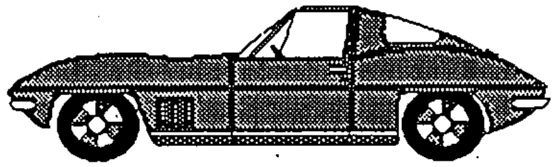
$\dot{x}$ – velocity, $\dot{x} = \frac{dx}{dt}$

$\ddot{x}$ – acceleration, $\ddot{x} = \frac{d^2x}{dt^2}$

Examples

1. A car is travelling along a straight road.
The distance, x (metres), travelled in t seconds, is $x = 10t - 5t^2$

Find its velocity when $t = 0{\cdot}5$ secs.



$$x = 10t - 5t^2$$
$$v = \frac{dx}{dt} = 10 - 10t$$

at $t = 0{\cdot}5$, $v = 10 - 5 = 5$ m/s.

2. A car is travelling along a straight road.
Its velocity, v (metres per second), in t seconds, is $v = 10 + 6t^2 - t^3$

Find the acceleration when t = 3 secs.

$$v = 10 + 6t^2 - t^3$$
$$a = \frac{dv}{dt} = 12t - 3t^2$$

at t = 3, a = 36 − 27 = 9 m/s^2

3. A car is travelling along a straight road.
Its distance, x (metres), travelled in t seconds, is $x = 5 + 2t + t^3$

Find the velocity and acceleration when t = 3 secs.

$$x = 5 + 2t + t^3$$
$$v = \frac{dx}{dt} = 2 + 3t^2, \quad at\ t = 3,\ v = 2 + 27 = 29 m/s$$
$$a = \frac{d^2x}{dt^2} = 6t, \quad at\ t = 3,\ a = 18\ m/s^2$$

Exercise 7

1. A body moves in a straight line and the motion is such that x, the number of metres from a fixed point after t secs, is given by

$$x = 3 - 4t + t^2.$$

(a) How far is the body from the fixed point at the start ?

(b) What is its position after 4 seconds ?

(c) What is its velocity after 3 seconds ?

(d) What is the initial acceleration ?

2. If $x = 4t^3 - 3t^2 - 2t - 1$, where x is in metres and t in seconds, find

(a) The velocity at the end of the 3rd and 4th seconds.

(b) The acceleration at the end of the 3rd and 4th seconds.

(c) The average velocity during the 4th second.

(d) The average acceleration during the 4th second.

3. A motor bike starts from rest and its displacement x m after t secs is given by:-

$$x = \frac{1}{6}t^3 + \frac{1}{4}t^2$$

Calculate the initial acceleration and the acceleration at the end of the 2nd second.

4. A body is moving in a straight line, so that after t seconds its displacement x metres from a fixed point O, is given by

$$x = 9t + 3t^2 - t^3.$$

(a) Find the initial displacement, velocity and acceleration of the body.

(b) Find the time at which the body is instantaneously at rest.

5. A body moves along a straight line so that after t seconds its displacement from a fixed point O on the line is x metres.
If $x = 3t^2(3 - t)$, find:–

(a) the initial velocity and acceleration.

(b) the velocity and acceleration after 3 seconds.

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

Page 370 Exercise 7.1:1.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 312 Exercise 12D.

Using the Second Derivative to determine the nature of Stationary Points

Consider the function

$f(x) = 2x^3 - 9x^2 + 12x$

$f'(x) = 6x^2 - 18x + 12$

For St. values $f'(x) = 0$

$6x^2 - 18x + 12 = 0$

$6(x^2 - 3x + 2) = 0$

$6(x - 1)(x - 2) = 0$

$x = 1$ or $x = 2$

$y = 5 \quad y = 4$

Therefore $f(x) = 2x^3 - 9x^2 + 12x$

has stationary points at (1,5) and(2,4).

Nature

x	<-	1	->	<-	2	->
$x - 1$	−	0	+	+	+	+
$x - 2$	−	−	−	−	0	+
$f'(x)$	+	0	−	−	0	+
$f(x)$	↗	→	↘	↘	→	↗

ie. Maximum T. Pt. at (1,5)

Minimum T. Pt. at (2,4)

Now consider the function defined by

$f'(x) = 6x^2 - 18x + 12$

$f''(x) = 12x - 18$

For St. values $f''(x) = 0$

$12x - 18 = 0$

$x = \frac{3}{2}$

$y = -\frac{3}{2}$

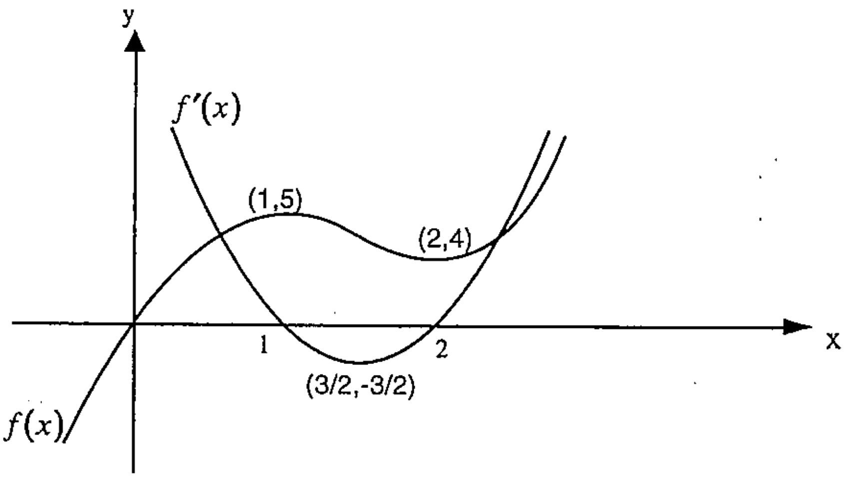
Therefore $f'(x) = 6x^2 - 18x + 12$ has a stationary point at $\left(\frac{3}{2}, -\frac{3}{2}\right)$

Nature

x	<-	$\frac{3}{2}$	->
$12x - 18$	−	0	+
$f''(x)$	−	0	+
$f'(x)$	↘	→	↗

ie. Minimum T. Pt. at $\left(\frac{3}{2}, -\frac{3}{2}\right)$

Now compare the graphs of $f(x)$ and $f'(x)$



The gradient of $f(x) = 2x^3 - 9x^2 + 12x$ is given by $f'(x)$.

The gradient of $f'(x) = 6x^2 - 18x + 12$ is given by $f''(x)$.

At the maximum turning point (1,5) , $f''(x)$ = negative. ie $f''(1)$ = – ve.

At the minimum turning point (2,4) , $f''(x)$ = positive. ie $f''(2)$ = +ve.

In general

If, at $x = a$,

(1) $f''(a) = +$ ve , then $f(x)$ has a minimum stationary value.

(2) $f''(a) = -$ ve , then $f(x)$ has a maximum stationary value,

(3) $f''(a) = 0$, then $f(x)$ has a possible point of inflexion , but use a table of signs to check.

At a point of inflexion , it can be shown that $f'(x) = 0$, a necessary but not sufficient condition.

For example; $f(x) = x^4 \Rightarrow f'(x) = 4x^3$

For S.V. $f'(x) = 0$ ie. $4x^3 = 0 \Rightarrow x = 0$ ie. St. Pt at (0,0)

Now $f''(x) = 12x^2$ and $f''(0) = 0$ -- a possible point of inflexion.

But consider the sign of $f'(x)$ at $x = 0$

x	<-	0	->
$f'(x)$	–	0	+
$f(x)$	↘	→	↗

ie. (0,0) is a minimum T.Pt.

Examples

1. Sketch the graph of the function $f(x) = (x+2)(x-1)^2$.

$f'(x) = (x-1)^2 + 2(x+2)(x-1)$ (using the product rule)

For S.V. $f'(x) = 0$

$(x-1)^2 + 2(x+2)(x-1) = 0$

$(x-1)[(x-1) + 2x + 4] = 0$

$3(x-1)(x+1) = 0$ (or $3x^2 - 3 = 0$)

i.e. $x = -1$ *and* $x = 1$

Stationary points at (–1,4) and (1,0)

$f''(x) = 6x$ [the 2nd derivative]

$f''(-1) =$ negative i.e. (–1,4) is a Maximum Turning point.

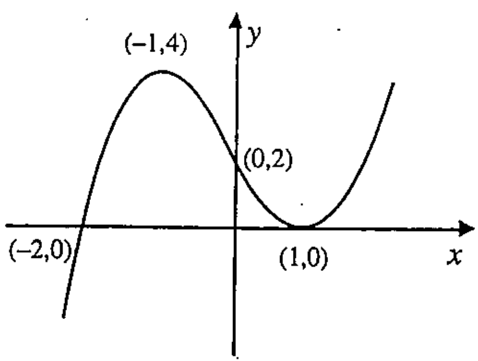
$f''(-1) =$ positive i.e. (1,0) is a Minimum Turning point.

When $f(x) = 0$, $(x+2)(x-1)^2 = 0 \Rightarrow x = -2$ and $x = 1$.

Curve crosses the x -axis at (–2,0) and (1,0)

When $x = 0$, $f(0) = 2$ i.e. y = 2.

Curve crosses the y -axis at (0,2)



2. Find the coordinates and nature of the stationary point on the curve

$$f(x) = x^3 - 81\ln x$$

$$f'(x) = 3x^2 - \frac{81}{x}$$

For S.V. $f'(x) = 0$

$$3x^2 - \frac{81}{x} = 0$$

$$3x^3 - 81 = 0 \Rightarrow x^3 = 27 \Rightarrow x = 3,\ y = 27 - 81\ln 3$$

$$f''(x) = 6x + \frac{81}{x^2}$$

$f''(3) = +ve$ i.e. $(3,\ 27 - 81\ln 3)$ is a Minimum Turning point.

3. Find the coordinates and nature of the stationary point on the curve

$$f(x) = e^x - 4x$$

$$f'(x) = e^x - 4$$

For S.V. $f'(x) = 0$

$$e^x - 4 = 0 \Rightarrow e^x = 4 \Rightarrow x = \ln 4, y = 4 - 4\ln 4$$

$$f''(x) = e^x$$

$f''(ln4) = +ve$ i.e. $(\ln 4, 4 - 4\ln 4)$ is a Min. Turning point.

Exercise 8

Use the second derivative to find the stationary values and their nature for the following functions.

1. $y = x - \ln x$
2. $y = x\ln x$
3. $y = xe^{-x}$
4. $y = \frac{1}{2}\sin\theta + \sin 2\theta$

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)

Page 61 Exercise 1.3:5. Questions 3,10(i) and Page 105 Exercise 2.5:2 Question 8 and Page 108 Exercise 2.5:4 Question 15

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 268 Exercise 10D Questions 9 - 14 and Page 480 Exercise 19A Questions 31,32,33.

Optimisation Problems

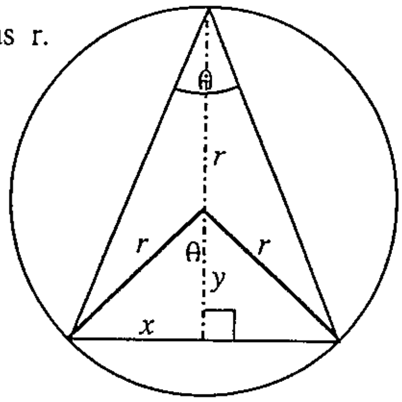
Example

1. An isosceles triangle is inscribed in a circle of radius r. Show that the area of the triangle is

$$A = r^2\sin\theta(1 + \cos\theta)$$

where θ is the angle between the equal sides.

Find the maximum possible area of the triangle.



From $\sin\theta = \frac{x}{r} \Rightarrow x = r\sin\theta$ and $\cos\theta = \frac{y}{r} \Rightarrow y = r\cos\theta$

The base is $2r\sin\theta$ and the height is $r + r\cos\theta$

$$\text{Area} = \tfrac{1}{2}\, 2r\sin\theta(r + r\cos\theta) = r^2\sin\theta(1 + \cos\theta)$$

$$A(\theta) = r^2\sin\theta(1 + \cos\theta) = r^2\sin\theta + r^2\sin\theta\cos\theta = r^2\sin\theta + \frac{1}{2}r^2\sin 2\theta$$

$A'(\theta) = r^2\cos\theta + r^2\cos 2\theta$

For S.V. $A'(\theta) = 0$

$r^2\cos\theta + r^2\cos 2\theta = 0$

$r^2\left(2\cos^2\theta + \cos\theta - 1\right) = 0 \Rightarrow r^2(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ and $\cos\theta = -1 \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = \pi$ (but $\theta \neq \pi$)

S.V. at $\theta = \frac{\pi}{3}$

$A''(\theta) = -r^2\sin\theta - 2r^2\sin 2\theta$

$A''\left(\frac{\pi}{3}\right) = -r^2\sin\frac{\pi}{3} - 2r^2\sin\frac{2\pi}{3} = -$ ve i.e. a maximum S.V. at $\theta = \frac{\pi}{3}$

The maximum area is

$$A\left(\frac{\pi}{3}\right) = r^2\sin\frac{\pi}{3}\left(1 + \cos\frac{\pi}{3}\right) = r^2\frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}r^2$$

Exercise 9

1. Four squares each of side s cm are cut from the corners of a metal square of side 16 cm. The metal is then bent to make an open topped tray of volume, V cm^3.

   (a) Prove that $V = 4s^3 - 64s^2 + 256s$.

   (b) Find the value of s which makes the volume a maximum.

2. A sector of a circle with radius r cm has an area of 16 cm^2.

(a) Show that the perimeter P cm of the sector is given by

$$P(r) = 2(r + \frac{16}{r})$$

(b) Find the minimum value of P.

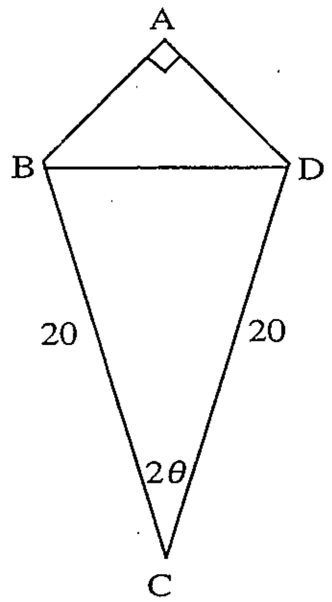
3. A cylindrical tank has a radius of r metres and a height of h metres.
The sum of the radius and the height is 2 metres.

(a) Prove that the volume, in m^3, is given by

$$V = \pi r^2(2 - r)$$

(b) Find the maximum volume.

4. ABCD is a kite which has AC as its axis of symmetry.
Angle BAD is right angled and BC and DC are 20 cm.



(a) Show that the area of triangle BCD is given by the expression $200\sin 2\theta$ and find an expression for BD^2.

(b) Use this expression for BD^2 to show that the area of triangle BAD is given by the expression $200 - 200\cos 2\theta$ and hence show that the area of the kite is given by the expression
$A(\theta) = 200(1 - \cos 2\theta + \sin 2\theta)$

(c) Find the value of θ which makes the area a maximum and find this maximum area.

Further examples can be found in the following resources.

The Complete A level Maths (Orlando Gough)
Examples can be found throughout most differentiation sections of trigonometric,exponential and logarithmic functions.

Understanding Pure Mathematics (A.J.Sadler/D.W.S.Thorning)

Page 268 Exercise 10D Questions 21 - 25.

Answers

Exercise 1 Page 11

1. $3x^2$ 2. $2x + 2$ 3. $6x + 4$

Exercise 2 Page 12

1. $3x^2 - 2x + 5$ 2. $6x + \frac{4}{x^2}$ 3. $\frac{1}{2x^{\frac{1}{2}}} - \frac{1}{2x^{\frac{3}{2}}}$ 4. $\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{2x^{\frac{3}{2}}}$

5. $-\frac{2}{x^3} + \frac{3}{x^4}$ 6. $-\frac{3}{2x^{\frac{5}{2}}} + \frac{3}{2}x^{\frac{1}{2}}$ 7. $20(4x + 5)^4$ 8. $\frac{4x^3}{(2x^4 - 3)^{\frac{1}{2}}}$

9. $\frac{3x}{(4 - x^2)^{\frac{3}{2}}}$ 10. $-\frac{4(x^2+1)}{(x^3+3x)^{\frac{4}{3}}}$ 11. $-3\sin x\cos^2 x$ 12. $\frac{\cos x}{2\sqrt{\sin x}}$

Exercise 3 Page 13

1. $(6x + 12)(x^2 + 4x - 5)^2$ 2. $\frac{3x^2}{2\sqrt{(x^3 + 5)}}$ 3. $\frac{4(1 + 2\sqrt{x})^3}{\sqrt{x}}$ 4. $\frac{3x}{(4 - x^2)^{\frac{3}{2}}}$

Exercise 4 Page 14

1. $2x(x - 3)(2x - 3)$ 2. $(8x + 3)(2x + 3)^2$ 3. $\frac{3(x - 4)}{2\sqrt{x - 6}}$ 4. $\frac{(x - 3)^2(7x - 3)}{2\sqrt{x}}$

5. $2(x + 1)(3x + 1)(x - 1)^3$ 6. $\frac{x^2(7x - 6)}{2\sqrt{x - 1}}$ 7. $\sin x + x\cos x$ 8. $x(2\sin x + x\cos x)$

9. $\cos 2x$ 10. $2\cos 2x\cos 5x - 5\sin 2x\sin 5x$

Exercise 5 Page 15

1. $\frac{x^2 + 6x}{(x + 3)^2}$ 2. $\frac{x - 8}{x^3}$ 3. $\frac{4(2x + 1)}{(1 - x)^4}$ 4. $\frac{2x(x - 4)}{(x - 2)^2}$

5. $-\frac{3(1 - 2x)^2}{x^4}$ 6. $-\frac{(3x + 4)}{2x^3\sqrt{x + 1}}$

Exercise 6 Page 18

1. $6\tan^2 2x\sec^2 2x$ 2. $8\mathrm{cosec}^4 x\cot x$ 3. $\sec x(\tan^2 x + \sec^2 x)$

4. $x(2\cot x - x\mathrm{cosec}^2 x)$ 5. $\frac{3}{3x + 2}$ 6. $-(x + 1)e^{-x}$

7. $\frac{(x + 1)e^x}{(x + 2)^2}$ 8. $\frac{x(2\ln x - 1)}{(\ln x)^2}$ 9. $\frac{x}{x^2 + 1}$

10. $e^{-2x^2}(1 - 4x^2)$ 11. $\frac{2}{1 - x^2}$ 12. $\frac{4}{(e^x + e^{-x})^2}$

Exercise 7 Page 20 - 21

1. (a) 3 m (b) 3 m (c) 2 m/s (d) 2 m/s^2

2. (a) 88 m/s, 166 m/s (b) 66 m/s2, 90 m/s^2
   (c) 127 m/s (d) 78 m/s^2

3. (a) $^1/_2$ m/s^2 (b) $2^1/_2$ m/s^2

4. (a) 0, 9 m/s, 6 m/s^2 (b) 3 secs

5. (a) 0 m/s, 18 m/s^2 (b) –27m/s, –36 m/s^2

Exercise 8 Page 24

1. Min at (1, 1) 2. Min at ($^1/_e$, $^1/_e$) 3. Max at (1,$^1/_e$)

4. Max at ($^\pi/_3$,$^{3\sqrt{3}}/_4$), P of I at (π, 0), Min at ($^{5\pi}/_3$,$-^{3\sqrt{3}}/_4$)

Exercise 9 Page 25 - 26

1. (a) Proof (b) $s = {}^8/_3$ cm

2. (a) Proof (b) P = 16 cm

3. (a) Proof (b) $V = {}^{32\pi}/_{27}$ cm3

4. (a) Proof (b) Proof
   (c) $\theta = {}^{3\pi}/_8$, A = 200(1 + √2)

Past Paper Questions

2001

Differentiate with respect to x $g(x)=e^{\cos 2x},\ o<x<\frac{\pi}{2}$. **(2 marks)**

2002

Given that $f(x)=\sqrt{x}e^{-x},\ x\geq 0$, obtain and simplify $f'(x)$. **(4 marks)**

2003

Given $f(x)=x(1+x)^{10}$, obtain $f'(x)$ and simplify your answer. **(3 marks)**

2004

Given $f(x)=\cos^2 xe^{\tan x}, \frac{-\pi}{2}<x<\frac{\pi}{2}$, obtain $f'(x)$ and evaluate $f'\left(\frac{\pi}{4}\right)$. **(3,1 marks)**

2005

(a) Given $f(x)=x^3\tan 2x$, where $0<x<\frac{\pi}{4}$, obtain $f'(x)$. **(3 marks)**

(b) For $y=\frac{1+x^2}{1+x}$, where $x\neq -1$, determine $\frac{dy}{dx}$ in simplified form. **(3 marks)**

2006

Differentiate, simplifying your answer: $\frac{1+\ln x}{3x}$, where $x>0$. **(3 marks)**

2007

Obtain the derivative of the function $f(x)=\exp(\sin 2x)$. **(3 marks)**

2009

Given $f(x)=(x+1)(x-2)^3$, obtain the values of x for which $f'(x)=0$. **(3 marks)**

2010

Differentiate the following functions

(a) $f(x)=e^x\sin x^2$. **(3 marks)**

(b) $g(x)=\dfrac{x^3}{1+\tan x}$. **(3 marks)**

2011

Given $f(x)=\sin x\cos^3 x$, obtain $f'(x)$. **(3 marks)**

2012

(a) Given $f(x)=\dfrac{3x+1}{x^2+1}$, obtain $f'(x)$. **(3 marks)**

(b) Let $g(x)=\cos^2 x\exp(\tan x)$. Obtain an expression for $g'(x)$ and simplify your answer. **(4 marks)**

2013

Differentiate $f(x)=e^{\cos x}\sin^2 x$. **(3 marks)**

2014

Given $f(x)=\dfrac{x^2-1}{x^2+1}$, obtain $f'(x)$ and simplify your answer. **(3 marks)**

2015

(a) For $y=\dfrac{5x+1}{x^2+2}$, find $\dfrac{dy}{dx}$. Express your answer as a single, simplified fraction. **(3 marks)**

(b) Given $f\left(x\right)=e^{2x}\sin^2 3x$, obtain $f'\left(x\right)$. **(3 marks)**