Further Calculus Past Papers Unit 3 Outcome 2

Multiple Choice Questions

Each correct answer in this section is worth two marks.

1. Differentiate $3 \cos(2x - \frac{\pi}{6})$ with respect to $x$.
   
   A. $-3 \sin(2x)$
   
   B. $-3 \sin(2x - \frac{\pi}{6})$
   
   C. $-6 \sin(2x - \frac{\pi}{6})$
   
   D. $6 \sin(2x - \frac{\pi}{6})$

   $\frac{d}{dx} \left(3 \cos(2x - \frac{\pi}{6})\right) = -3 \cdot 2 \sin(2x - \frac{\pi}{6})$

   $= -6 \sin(2x - \frac{\pi}{6})$

   Option C

[END OF MULTIPLE CHOICE QUESTIONS]

Written Questions

2. Differentiate $\sin 2x + \frac{2}{\sqrt{x}}$ with respect to $x$.

   $\frac{d}{dx} \left(\sin 2x + \frac{2}{\sqrt{x}}\right) = 2 \cos 2x - \frac{1}{x^{3/2}}$

   $\frac{d}{dx} \left(\sin 2x\right) = 2 \cos 2x$

   $\frac{d}{dx} \left(\frac{2}{\sqrt{x}}\right) = -\frac{1}{x^{3/2}}$

   $\frac{d}{dx} \left(\sin 2x + \frac{2}{\sqrt{x}}\right) = 2 \cos 2x - \frac{1}{x^{3/2}}$
3. Given that \( f(x) = (5x - 4)^{\frac{3}{2}} \), evaluate \( f'(4) \).

<table>
<thead>
<tr>
<th>Part</th>
<th>Marks</th>
<th>Level</th>
<th>Calc.</th>
<th>Content</th>
<th>Answer</th>
</tr>
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<td>CN</td>
<td>C21</td>
<td>( \frac{3}{8} )</td>
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</tr>
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<td>A/B</td>
<td>CN</td>
<td>C21</td>
<td></td>
<td>2000 P2 Q8</td>
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</tbody>
</table>

- \( ^1 \) pd: differentiate power
- \( ^2 \) pd: differentiate 2nd function
- \( ^3 \) pd: evaluate \( f'(x) \)

\[ f'(4) = \frac{3}{8} \]

4. Given \( f(x) = \cos^2 x - \sin^2 x \), find \( f'(x) \).

<table>
<thead>
<tr>
<th>Part</th>
<th>Marks</th>
<th>Unit</th>
<th>non-calc</th>
<th>calc</th>
<th>calc neat</th>
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<td>2</td>
<td></td>
<td>3.2.2</td>
<td></td>
</tr>
</tbody>
</table>

- \( ^1 \) \( f(x) = \cos 2x \)
- \( ^2 \) \( -\sin 2x \)
- \( ^3 \) \( \times 2 \)

\[ f'(x) = 2 \cos x \times -\sin x \]

5. Given that \( f(x) = 5(7 - 2x)^3 \), find the value of \( f'(4) \).

<table>
<thead>
<tr>
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<td>A/B</td>
<td>A/B</td>
<td></td>
<td>3.2.2</td>
<td></td>
</tr>
</tbody>
</table>

- \( ^1 \) \( (7 - 2x)^2 \)
- \( ^2 \) \( \times 15 \)
- \( ^3 \) \( \times -2 \)
- \( ^4 \) \( -30 \)

\[ f'(4) = \frac{3}{8} \]
6. Differentiate \(2x^3 + \sin^2 x\) with respect to \(x\).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{part marks} & \text{Unit} & \text{non-calc} & \text{calc} & \text{calc neut} & \text{Content Reference} : & \text{3.2} \\
\hline
. & 4 & 3.2 & 1 & 3 & & \\
\hline
\end{array}
\]

\[3x^2\]
\[2 \sin x \cdot \cos x\]
\[(\sin x)^2\] stated or implied by \(\cdot^3\)
\[2 \sin x\]
\[\cdot^4 \times \cos x\]

7. Find the derivative, with respect to \(x\), of \(\frac{1}{x^3} + \cos 3x\).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{part marks} & \text{Unit} & \text{non-calc} & \text{calc} & \text{calc neut} & \text{Content Reference} : & \text{3.2} \\
\hline
. & 4 & 3.2 & 4 & & & \\
\hline
\end{array}
\]

\[x^{-3}\] stated or implied by \(\cdot^2\)
\[3x^{-4}\]
\[-\sin 3x\]
\[\times 3\]

8. If \(f(x) = \cos^2 x - \frac{2}{3x^2}\), find \(f'(x)\).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{part marks} & \text{Unit} & \text{non-calc} & \text{calc} & \text{calc neut} & \text{Content Reference} : & \text{3.2} \\
\hline
. & 4 & 3.2 & 2 & 2 & & \\
\hline
\end{array}
\]

\[-\frac{3}{2} x^{-2}\]
\[2 \cos x\]
\[x (-\sin x)\]
\[\frac{4}{3} x^{-3}\]
9. Differentiate $4\sqrt{x} + 3\cos 2x$ with respect to $x$.

<table>
<thead>
<tr>
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<th>non-calc</th>
<th>A/B</th>
<th>calc</th>
<th>A/B</th>
<th>calc neut</th>
<th>A/B</th>
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<td>3.2.2</td>
<td>1.34</td>
<td>Source:</td>
<td>1993 P1 qu.9</td>
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</table>

\[
\begin{align*}
\text{\textbullet}^1 & \quad 4x \\
\text{\textbullet}^2 & \quad 3 \cos 2x \\
\text{\textbullet}^3 & \quad -2 \sin 2x \\
\text{\textbullet}^4 & \quad 2
\end{align*}
\]

10. Find $\frac{dy}{dx}$ given that $y = \sqrt{1 + \cos x}$.

<table>
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<th>A/B</th>
<th>calc</th>
<th>A/B</th>
<th>calc neut</th>
<th>A/B</th>
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<tbody>
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<td>3</td>
<td>3</td>
<td>3.2.2</td>
<td>3.2.1</td>
<td>Source:</td>
<td>1996 P1 qu.13</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{\textbullet}^1 & \quad \left(1 + \cos x\right)^{\frac{3}{2}} \text{ stated or implied by } \text{\textbullet}^2 \\
\text{\textbullet}^2 & \quad \frac{1}{2} \left(1 + \cos x\right)^{-\frac{1}{2}} \\
\text{\textbullet}^3 & \quad x - \sin x
\end{align*}
\]

11. Given $f(x) = (\sin x + 1)^2$, find the exact value of $f'(\frac{\pi}{6})$.

<table>
<thead>
<tr>
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<th>non-calc</th>
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<th>calc</th>
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<th>calc neut</th>
<th>A/B</th>
<th>Content Reference:</th>
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<td>3.2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3.2</td>
<td>3.2.1</td>
<td>3.2.2</td>
<td>Source:</td>
<td>1998 P1 qu.16</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{\textbullet}^1 & \quad 2(\sin x + 1) \\
\text{\textbullet}^2 & \quad \cos x \\
\text{\textbullet}^3 & \quad \frac{3\sqrt{3}}{2}
\end{align*}
\]

\[
\begin{align*}
\text{\textbullet}^1 & \quad \text{expand and differentiate } 2\sin x + 1 \\
\text{\textbullet}^2 & \quad \text{differentiate } \sin^2 x \\
\text{\textbullet}^3 & \quad \frac{3\sqrt{3}}{2}
\end{align*}
\]
12. Find the equation of the tangent to the curve \( y = 2 \sin \left( x - \frac{\pi}{6} \right) \) at the point where \( x = \frac{\pi}{3} \).

\[
\frac{dy}{dx} = 2 \cos \left( x - \frac{\pi}{6} \right)
\]
\[
m = \sqrt{3}
\]
\[
y_{x=\frac{\pi}{3}} = 1
\]
\[
y - 1 = \sqrt{3} \left( x - \frac{\pi}{3} \right)
\]

13. Find \( \int \sqrt{1 + 3x} \, dx \) and hence find the exact value of \( \int_{0}^{1} \sqrt{1 + 3x} \, dx \).

\[
\int (1 + 3x)^{\frac{1}{2}} \, dx = \frac{1}{2} (1 + 3x)^{\frac{3}{2}}
\]

14. Differentiate \( \sin^3 x \) with respect to \( x \).

Hence find \( \int \sin^2 x \cos x \, dx \).
15. Find \( \int \frac{1}{(7-3x)^2} \, dx \).

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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>A/B</td>
<td>CN</td>
<td>C22, C14</td>
<td>( \frac{1}{3(7-3x)} + c )</td>
</tr>
</tbody>
</table>

- \( 1 \) \( \text{pd: integrate function} \)
- \( 2 \) \( \text{pd: deal with function of function} \)

16. Evaluate \( \int_{-3}^{0} (2x + 3)^2 \, dx \).

17. (a) Evaluate \( \int_{0}^{\pi} \cos 2x \, dx \).

(b) Draw a sketch and explain your answer.
[SQA] 18.

(a) Show that \((\cos x + \sin x)^2 = 1 + \sin 2x\).

(b) Hence find \(\int (\cos x + \sin x)^2 \, dx\).

18.

\[ \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{part} & \text{marks} & \text{Unit} & \text{non-calc} & \text{calc} & \text{calc neut} & \text{Content Reference :} & \text{3.2} \\
\hline
(a) & 1 & \text{2.3} & \text{3} & \text{1} & \text{3} & 2.3.3 & \text{Source} \\
(b) & 3 & \text{3.2} & \text{ } & \text{ } & \text{ } & 3.2.4 & \text{1993 P1 qu.19} \\
\hline
\end{array} \]

19. Find \(\int \left(6x^2 - x + \cos x\right) \, dx\).

19.

\[ \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{part} & \text{marks} & \text{Unit} & \text{non-calc} & \text{calc} & \text{calc neut} & \text{Content Reference :} & \text{3.2} \\
\hline
1 & 4 & \text{3.2} & \text{4} & \text{ } & \text{ } & 3.2.4 & \text{Source} \\
\hline
\end{array} \]

Questions marked ‘[SQA]’ © SQA
All others © Higher Still Notes
20.  
(a) By writing $\sin 3x$ as $\sin(2x + x)$, show that $\sin 3x = 3\sin x - 4\sin^3 x$.  

(b) Hence find $\int \sin^3 x \, dx$.  

---

21.  
(a) Find the derivative of the function $f(x) = (8 - x^3)^{\frac{1}{2}}$, $x < 2$.  

(b) Hence write down $\int \frac{x^2}{(8 - x^3)^{\frac{3}{2}}} \, dx$.  

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<tbody>
<tr>
<td>(a)</td>
<td>4</td>
<td>A/B</td>
<td>CN</td>
<td>C21</td>
<td>$\frac{1}{2}x^2(8 - x^3)^{-\frac{3}{2}}$</td>
<td>3.2, 3.3</td>
<td>1995 Paper 2 Qu.9</td>
</tr>
<tr>
<td>(b)</td>
<td>4</td>
<td>A/B</td>
<td>CN</td>
<td>C24</td>
<td>$-\frac{2}{3}(8 - x^3)^{\frac{3}{2}} + c$</td>
<td>3.2.4, 3.2.5</td>
<td>2002 P1 Q10</td>
</tr>
</tbody>
</table>

- 1 pd: process differentiation  
- 2 pd: use the chain rule  
- 3 ic: interpret answer from (a)
22. The curve \( y = f(x) \) passes through the point \( \left( \frac{\pi}{12}, 1 \right) \) and \( f'(x) = \cos 2x \).

Find \( f(x) \).

\[
\begin{align*}
1 & \frac{1}{2} \sin 2x \\
2 & 1 = \frac{3}{2} \sin \frac{\pi}{3} + c \\
3 & c = \frac{3}{4}
\end{align*}
\]

23. The graph of \( y = f(x) \) passes through the point \( \left( \frac{\pi}{9}, 1 \right) \).

If \( f'(x) = \sin(3x) \) express \( y \) in terms of \( x \).

\[
\begin{align*}
1 & \text{ ss: know to integrate} \\
2 & \text{ pd: integrate} \\
3 & \text{ ic: interpret } \left( \frac{\pi}{9}, 1 \right) \\
4 & \text{ pd: process}
\end{align*}
\]

\[
1 \ y = \int \sin(3x) \, dx \quad \text{stated or implied by} \\
2 \ - \frac{1}{3} \cos(3x) \\
3 \ 1 = -\frac{1}{3} \cos \left( \frac{\pi}{3} \right) + c \text{ or equiv.} \\
4 \ c = \frac{7}{6}
\]

24. A curve for which \( \frac{dy}{dx} = 3 \sin(2x) \) passes through the point \( \left( \frac{5\pi}{12}, \sqrt{3} \right) \).

Find \( y \) in terms of \( x \).

\[
\begin{align*}
1 & \text{ pd: integrate trig function} \\
2 & \text{ pd: integrate composite function} \\
3 & \text{ ss: use given point to find } "c" \\
4 & \text{ pd: evaluate } "c"
\end{align*}
\]

\[
1 \ \int 3 \sin(2x) \, dx \quad \text{stated or implied by} \\
2 \ - \frac{3}{2} \cos(2x) \\
3 \ \sqrt{3} = -\frac{3}{2} \cos(2 \times \frac{5}{12} \pi) + c \\
4 \ c = \frac{1}{4} \sqrt{3} (\approx 0.4)
\]
25. A point moves in a straight line such that its acceleration $a$ is given by $a = 2(4 - t)^{\frac{1}{2}}$, $0 \leq t \leq 4$. If it starts at rest, find an expression for the velocity $v$ where $a = \frac{dv}{dt}$.

<table>
<thead>
<tr>
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<th>Answer</th>
<th>U3 OC2</th>
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</thead>
<tbody>
<tr>
<td>4</td>
<td>C</td>
<td>NC</td>
<td>C18, C22</td>
<td>$V = -\frac{2}{3}(4 - t)^{\frac{3}{2}} + \frac{32}{3}$</td>
<td>2002 P2 Q8</td>
<td></td>
</tr>
</tbody>
</table>

- $1$ ss: know to integrate acceleration
- $2$ pd: integrate
- $3$ ic: use initial conditions with const. of int.
- $4$ pd: process solution

- $1$ $V = \int (2(4 - t)^{\frac{1}{2}}) \, dt$ stated or implied by $2$
- $2$ $2 \times \frac{1}{2}(4 - t)^{\frac{3}{2}}$
- $3$ $0 = 2 \times \frac{1}{2}(4 - 0)^{\frac{3}{2}} + c$
- $4$ $c = 10\frac{2}{3}$

26. A sketch of part of the graph of $y = \sin 2x$ is shown in the diagram.

The points $P$ and $Q$ have coordinates $(p, 0)$ and $(q, -1)$.
(a) Write down the values of $p$ and $q$.
(b) Find the area of the shaded region.

- $1$ $p = \frac{\pi}{2}$ and $q = \frac{3\pi}{2}$
- $2$ $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\sin 2x) \, dx$
- $3$ $-\frac{1}{2} \cos 2x$
- $4$ $-\frac{1}{2}$
- $5$ deal with – we correctly giving $\frac{1}{2}$
27. An artist has designed a 'bow' shape which he finds can be modelled by the shaded area below. Calculate the area of this shape.

\[ y = \cos 2x \]

---

**Questions marked `[SQA]` © SQA  
All others © Higher Still Notes**
28. Linktown Church is considering designs for a logo for their Parish magazine. The ‘C’ is part of a circle and the centre of the circle is the mid-point of the vertical arm of the ‘L’. Since the ‘L’ is clearly smaller than the ‘C’, the designer wishes to ensure that the total length of the arms of the ‘L’ is as long as possible.

The designer decides to call the point where the ‘L’ and ‘C’ meet A and chooses to draw co-ordinate axes so that A is in the first quadrant. With axes as shown, the equation of the circle is \( x^2 + y^2 = 20 \).

(a) If A has co-ordinates \((x,y)\), show that the total length \( T \) of the arms of the ‘L’ is given by \( T = 2x + \sqrt{20-x^2} \).

(b) Show that for a stationary value of \( T \), \( x \) satisfies the equation \( x = 2\sqrt{20 - x^2} \).

(c) By squaring both sides, solve this equation. Hence find the greatest length of the arms of the ‘L’.

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<table>
<thead>
<tr>
<th>part</th>
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<th>Unit</th>
<th>non-calc C</th>
<th>calc A/B</th>
<th>calc neut C</th>
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<td>1</td>
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<td>(c)</td>
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<td>1.3</td>
<td>1</td>
<td></td>
<td>1</td>
<td>2</td>
<td>1.3,15</td>
<td></td>
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</tbody>
</table>

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(a) \( T = x + x + y \) and \( y^2 = 20 - x^2 \)

(b) \( x^2 \) appearance of \( \frac{dT}{dx} = -2 \) + .......

(c) \( x = 4 \) (accept \( x = \pm 4 \))

---

\[ x = 4 \text{ giving } T_{\text{max}} = 10 \]
29. An oil production platform, 9√3 km offshore, is to be connected by a pipeline to a refinery on shore, 100 km down the coast from the platform as shown in the diagram.

The length of underwater pipeline is \( x \) km and the length of pipeline on land is \( y \) km. It costs £2 million to lay each kilometre of pipeline underwater and £1 million to lay each kilometre of pipeline on land.

(a) Show that the total cost of this pipeline is \( LC(x) \) million where

\[ C(x) = 2x + 100 - \left( x^2 - 243 \right)^{\frac{1}{2}}. \]

(b) Show that \( x = 18 \) gives a minimum cost for this pipeline.

Find this minimum cost and the corresponding total length of the pipeline.

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<td>7</td>
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<td>1</td>
<td>2</td>
<td>1.3.15, 3.2.2</td>
<td></td>
</tr>
</tbody>
</table>

(a) \( C = 2x + y \)
(b) \( \frac{1}{2} \left( x^2 - (9\sqrt{3})^2 \right) \)

for completing proof

(b) knowing to differentiate

\( \frac{1}{2} \left( x^2 - 243 \right)^{-\frac{1}{2}} \)

\( \times \ 2x \)

\( C'(18) = 0 \)

justification of minimum e.g. nature table

\( C = 127 \)

\( x + y = 109 \)

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[END OF WRITTEN QUESTIONS]