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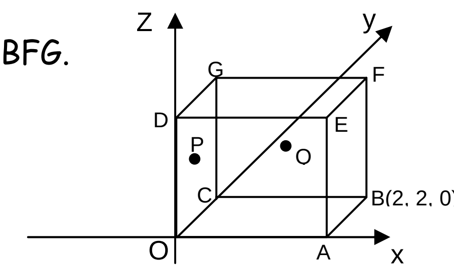
1. Given the cube with sides 2 units and B is (2,2,0).
Also P and Q are the centres of face OCGD and CBFG.



(a) Coordinates og G are (0, 2, 2)

(b) Position vectors of p and q are:

$$\underline{p} = 0\underline{i} + \underline{j} + \underline{k} \quad \text{and} \quad \underline{q} = \underline{i} + 2\underline{j} + \underline{k}$$

(c) Size of angle POQ is:

$$\cos\theta = \frac{\underline{p}\cdot\underline{q}}{|\underline{p}|\cdot|\underline{q}|} \qquad \underline{p}\cdot\underline{q} = (0,1,1)\cdot(1,2,1) = 3$$

$$= \frac{3}{\sqrt{2}\cdot\sqrt{6}} \qquad \underline{p} = \sqrt{0^2+1^2+1^2} = \sqrt{2}$$

$$= \frac{\sqrt{3}}{2} \qquad \underline{q} = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\theta = 30^o$$

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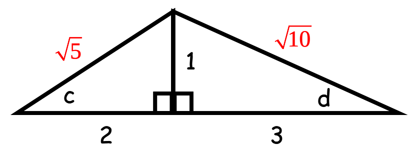
2. Given the diagram.



(a) The exact value of $\sin(c+d)$

$$\sin(c+d) = \sin c\cos d + \sin d\cos c$$
$$= \frac{1}{\sqrt{5}}\cdot\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}}\cdot\frac{2}{\sqrt{5}}$$
$$= \frac{3}{5\sqrt{2}} + \frac{2}{5\sqrt{2}}$$
$$= \frac{5}{5\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}$$

(b)(i) The exact value of $\sin 2c$:

$$\sin 2c = 2\cdot\sin c\cos c$$
$$= 2\cdot\frac{1}{\sqrt{5}}\cdot\frac{2}{\sqrt{5}}$$
$$= \frac{4}{5}$$

(b)(ii) The value of $\cos 2d$:

$$\cos 2d = \cos^2 d - \sin^2 d$$
$$= \left(\frac{3}{\sqrt{10}}\right)^2 - \left(\frac{1}{\sqrt{10}}\right)^2$$
$$= \frac{9}{10} - \frac{1}{10}$$
$$= \frac{8}{10}$$
$$= \frac{4}{5}$$

Hence $\cos 2d$ is exactly the same as $\sin 2c$

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3. To show that the line $y = 6 - 2x$ is a tangent to the circle $x^2 + y^2 + 6x - 4y - 7 = 0$

sub $y = 6 - 2x$ into the circle equation

$$x^2 + (6 - 2x)^2 + 6x - 4(6 - 2x) - 7 = 0$$
$$x^2 + 4x^2 - 24x + 36 + 6x - 24 + 8x - 7 = 0$$
$$5x^2 - 10x + 5 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)(x - 1) = 0 \qquad x = 1$$

If a tangent then

$$b^2 - 4ac = 0$$
$$(-2)^2 - 4 \times 1 \times 1 = 0$$
$$4 - 4 = 0$$
$$0 = 0$$

Coordinate is

$$x = 1 \quad y = 6 - 2 \times 1 = 4 \qquad (1,4)$$

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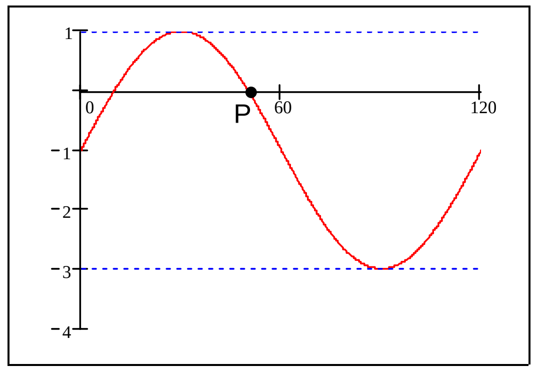
4. Given the diagram and the function has the form $y = a\sin(bx^o) + c$



(a) a = 2 b = 3 c = -1

(b) The exact value of the x-coordinate of P is given by:

$$2\sin 3x - 1 = 0$$
$$2\sin 3x = 1$$
$$\sin 3x = \frac{1}{2}$$
$$3x = \sin^{-1}\left(\frac{1}{2}\right)$$
$$3x = 30, 150, (30+360), (150+360), ...$$
$$3x = 30, 150, 390, 510, ...$$
$$x = 10, 50, 130, 170, ...$$

From the graph it is clear x - cordinate is 50°

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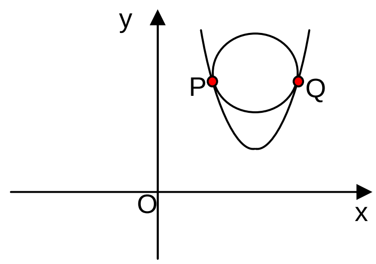
5. Given the diagram. The circle touches the parabola with equation

$$y = \frac{1}{2}x^2 - 8x + 34$$

at the points P and Q.



(a) Given the gradient at Q is 4.
The coordinates of Q are:

Gradient is $y'(x) = x - 8$

$$4 = x - 8$$
$$x - 8 = 4$$
$$x = 12$$

y coordinate $= \frac{1}{2} \times 12^2 - 8 \times 12 + 34 = 10$

Coordinates for P (12,10)

(b) The coordinates for P:

By symmetry of the parabola and circle

Gradient is $y'(x) = x - 8$

$$-4 = x - 8$$
$$x - 8 = -4$$
$$x = 4$$

Coordinates for P (4,10)

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(c) The coordinates of *C* are given when the lines perpendicular to each tangent P and Q intersect:

Line perpendicular to P(4,10) $y - 10 = \frac{1}{4}(x - 4)$

$4y - 40 = x - 4$

$4y = x + 36$

Line perpendicular to P(12,10) $y - 10 = -\frac{1}{4}(x - 12)$

$4y - 40 = -x + 12$

$4y = -x + 52$

x coordinate of *C* is $x + 36 = -x + 52$

$2x = 16$

$x = 8$

y coordinate is given by $4y = 8 + 36$

$y = \frac{44}{4} = 11$

Coordinates for P (8,11)

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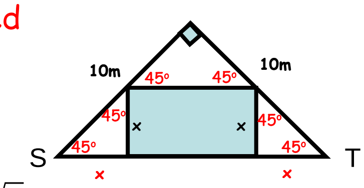
6. Given the diagram of the garden in the shape of the right-angled isosceles triangle. Red values have been added



(a)(i) The exact value of ST:

$$ST = \sqrt{10^2 + 10^2} = \sqrt{200} = \sqrt{100} \cdot \sqrt{2} = 10\sqrt{2}$$

(ii) Area of the decking is:

$$Area = length \times breadth$$
$$= (10\sqrt{2} - x - x) \times x$$
$$= (10\sqrt{2} - 2x)x$$
$$= 10\sqrt{2}x - 2x^2$$

(b) Dimensions that give maximum area of decking are given by:

$$\frac{dA}{dx} = 10\sqrt{2} - 4x$$

Maximum is given by

$$\frac{dA}{dx} = 10\sqrt{2} - 4x = 0$$

$$4x = 10\sqrt{2}$$
$$x = \frac{10\sqrt{2}}{4} = \frac{5\sqrt{2}}{2}$$

Hence dimensions are

$$length = 10\sqrt{2} - 2 \times \frac{5\sqrt{2}}{2} = 5\sqrt{2}$$

$$breadth = \frac{5\sqrt{2}}{2}$$

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7. The value of

$$y = \int_0^2 \sin(4x+1)dx$$

$$y = \left[-\frac{1}{4}\cos(4x+1) \right]_0^2$$

$$y = -\frac{1}{4}\left[\cos(9) - \cos(1)\right]$$

$$y = 0.363$$

8. Given the equation $y = \log_3(x-1) - 2.2$, where x > 1, cuts the x – axis at the point (a,0).

   The value of a is:

$$0 = \log_3(a-1) - 2.2$$

$$2.2 = \log_3(a-1)$$

$$3^{2.2} = a - 1$$

$$3^{2.2} + 1 = a$$

$$a = 12.21$$

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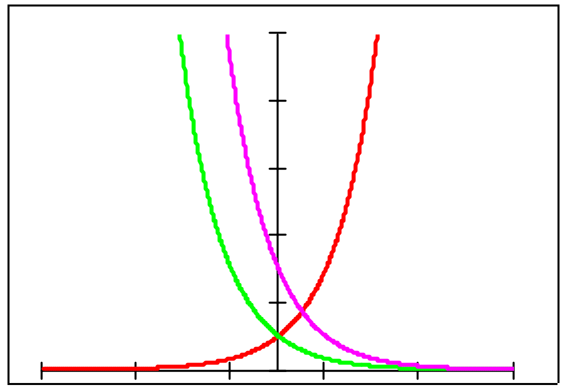
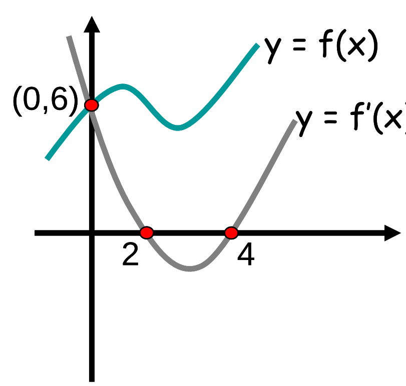
9. Given the graph of $y = a^x$, $a > 1$



(a) Graph of $y = a^{-x}$
Green line
Reflected in y – axis

(b) Graph of $y = a^{1-x}$
Pink line
Move $y = a^{-x}$ 1 unit to the right.

10. Given the diagram of the cubic function and its derived function.
Both pass through (0,6) and the derived function passes through the points (2,0) and (4,0).



(a) Given that $f'(x)$ is of the form $k(x-a)(x-b)$:

(i) Values of a and b are 2 and 4.

(ii) The value of k is:

$$f'(x) = k(x-a)(x-b)$$
$$6 = k(0-2)(0-4)$$
$$6 = 8k$$
$$k = \frac{6}{8} = \frac{3}{4}$$

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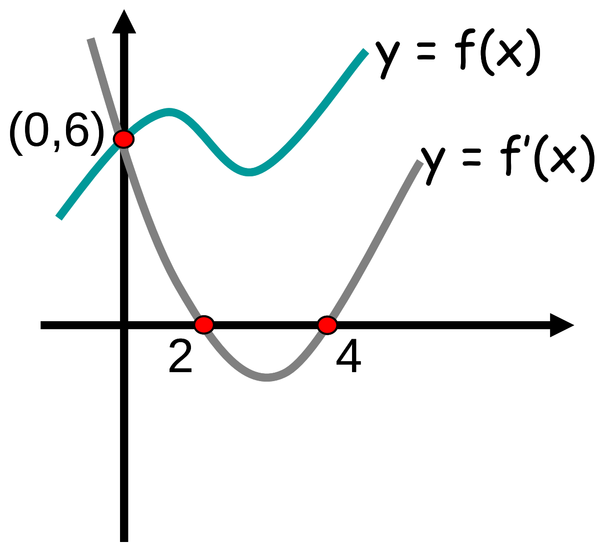
10. (b) Equation of cubic is:



$$f'(x) = \frac{3}{4}(x-2)(x-4)$$
$$= \frac{3}{4}(x^2 - 6x + 8)$$
$$= \frac{3}{4}x^2 - \frac{9}{2}x + 6$$

$$f(x) = \int\left[\frac{3}{4}x^2 - \frac{9}{2}x + 6\right]dx$$
$$= \frac{1}{4}x^3 - \frac{9}{4}x^2 + 6x + c$$

for $x = 0 \quad f(x) = 6$

Cubic has equation $f(x) = \frac{1}{4}x^3 - \frac{9}{4}x^2 + 6x + 6$

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11. Given x and y satisfy the equation $y = 3 \times 4^x$

(a) The value of a if (a,6) lies of the graph equation $y = 3 \times 4^x$:

$$6 = 3 \times 4^a$$
$$\frac{6}{3} = 4^a$$
$$2 = 4^a$$
$$\log_4 2 = a$$
$$a = \log_4 4^{0.5}$$
$$a = 0.5 \qquad \log_4 4 = 1$$

(b) The value of b if (-0.5,b) lies of the graph equation $y = 3 \times 4^x$:

$$b = 3 \times 4^{-\frac{1}{2}}$$
$$b = 3 \times \frac{1}{4^{\frac{1}{2}}}$$
$$b = 3 \times \frac{1}{2}$$
$$b = \frac{3}{2}$$

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11. (c) Given a graph of $\log_{10} y$ against x is drawn.

$y = 3 \times 4^x$

$\log_{10} y = \log_{10} 3 \times 4^x$

$\log_{10} y = \log_{10} 3 + \log_{10} 4^x$

$\log_{10} y = \log_{10} 3 + x\log_{10} 4$

$\log_{10} y = x(\log_{10} 4) + \log_{10} 3$

Comparing to $\log_{10} y = Px + Q$

Gradient is $\log_{10} 4$