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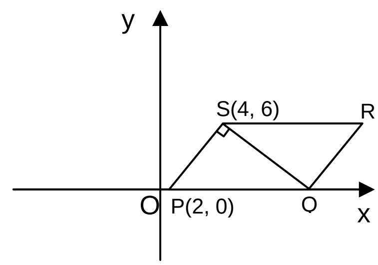
c1. Given the parallelogram PQRS.



(a) The equation of QS is:

Gradient $m_{PS} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-0}{4-2} = 3$

Using the perpendicular formula

$m_{QS} = -\frac{1}{3}$

Hence equation of QS is

$y - b = m(x - a)$ S$(4,6)$

$y - 6 = -\frac{1}{3}(x-4)$

$3y - 18 = -x + 4$

$x + 3y = 22$ as required.

(b) For coordinates of Q and R we have:

When $y = 0$

$x + 3\cdot 0 = 22$

Q is the point (22,0)

By the symmetry of a paralleogram

$R = (22,0) + (2,6) = (24,6)$

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2. Given $kx^2 + kx + 6 = 0$, $k \neq 0$. For equal roots we have:

Compare to standard quadratic

$$\begin{array}{l} ax^2 + bx + c = 0 \\ \downarrow \quad\;\; \downarrow \quad \downarrow \\ kx^2 + kx + 6 = 0 \\ a = k \\ b = k \\ c = 6 \end{array}$$

For equal roots we need:

$$\begin{aligned} b^2 - 4ac &= 0 \\ k^2 - 4 \times k \times 6 &= 0 \\ k(k - 24) &= 0 \\ k &= 24 \end{aligned}$$

$$\text{Note that } k = 0 \text{ not valid since } k > 0$$

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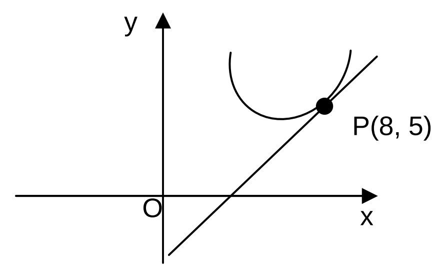
3. Given parabola $y = x^2 - 14x + 53$ and tangent point P(8,5).

(a) The equation of the tangent is:



$$y'(x) = 2x - 14$$

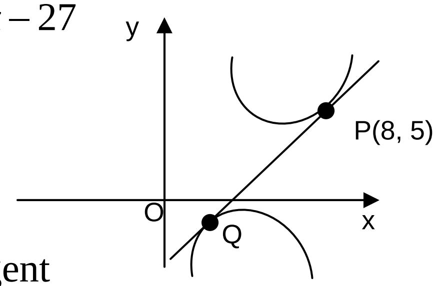
Gradient is $y'(8) = 2 \times 8 - 14 = 2$

equation of tangent is:

$$y - b = m(x - a) \quad \text{Pt is } (8,5)$$
$$y - 5 = 2(x - 8)$$
$$y = 2x - 11$$

(b) To show that equation in (a) is also a tangent to

$$y = -x^2 + 10x - 27$$



Sub $y = 2x - 11$ into $y = -x^2 + 10x - 27$

$2x - 11 = -x^2 + 10x - 27$

$x^2 + 8x - 16 = 0$

$(x-4)^2 = 0$

Since equal roots then line is tangent to the parabola.

The point of contact is:

When $x = 4$ $\quad y = 2(4) - 11 = -3$ $\quad$ Contact Pt. is Q(4,-3)

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4. Given the two circles have the same centre. The small and large circles have equations:

$$(x-3)^2+(y-4)^2=25 \qquad x^2+y^2-kx-8y-2k=0$$

(a) Radius of larger circle is:

Using equation of the small circle centre is: $(3,4)$.

From the larger circle: $C\left(\frac{1}{2}k,4\right) \quad \Rightarrow k=6$

Hence radius of larger circle is:

$$\begin{aligned} r &= \sqrt{g^2+f^2-c} \\ &= \sqrt{(-3)^2+(-4)^2-(-12)} \\ &= \sqrt{37} \end{aligned}$$

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5. Given the curve $y = f(x)$ is such that $\frac{dy}{dx} = 4x - 6x^2$.

Also the curve passes through the point (-1,9).

Then y in terms of x is:

$$y = \int (4x - 6x^2)dx = 2x^2 - 2x^3 + c$$

Since curve passes through (-1,9) we have

$$9 = 2(-1)^2 - 2(-1)^3 + c$$

$$9 = 2 + 2 + c$$

$$c = 5$$

$$y = 2x^2 - 2x^3 + 5$$

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6. Given the points P(-1,2,-1) and Q (3,2,-4)

(a) $\overrightarrow{PQ}$ in component form is: $\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

(b) The length of $\overrightarrow{PQ}$ is: $\sqrt{4^2 + 0^2 + (-3)^2} = \sqrt{25} = 5$

(c) A component vector parallel to $\overrightarrow{PQ}$ is: $8\underline{i} + 0\underline{j} - 6\underline{k}$

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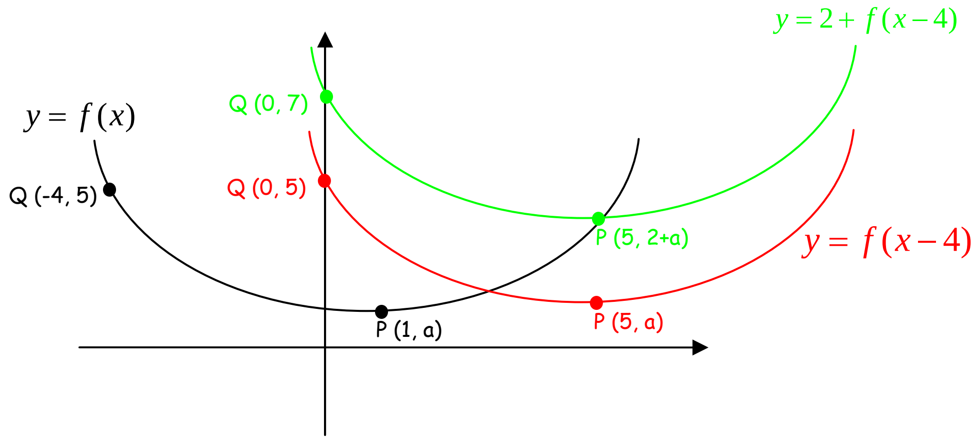
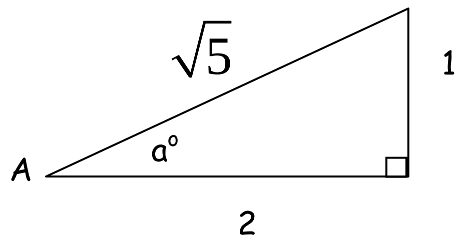
7. Given the graph of the $y = f(x)$

(a) The graph of $y = f(x-4)$ is:

(b) The graph of $y = 2 + f(x-4)$ is:



8. Given the triangle diagram.



(a)(i) The exact value of $\sin a^o$ is:

$$\sin a^o = \frac{opp}{hyp} = \frac{1}{\sqrt{5}}$$

(ii) The exact value of $\sin 2a^o$ is:

$$\sin 2a^o = 2\sin a^o \cos a^o = 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{4}{5}$$

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8. (b) The exact value of $\sin 3a^o$ is:

$$\sin 3a^o = \sin(2a^o + a^o) = \sin 2a^o \cos a^o + \cos 2a \sin a^o$$
$$= \sin 2a^o \cdot \cos a^o + (\cos^2 a - \sin^2 a) \cdot \sin a^o$$
$$= \frac{4}{5} \cdot \frac{2}{\sqrt{5}} + \left(\frac{4}{5} - \frac{1}{5}\right) \cdot \frac{1}{\sqrt{5}}$$
$$= \frac{8}{5\sqrt{5}} + \frac{3}{5\sqrt{5}}$$
$$= \frac{11}{5\sqrt{5}}$$

9. Given $y = \frac{1}{x^3} - \cos 2x$, $x \neq 0$ then $\frac{dy}{dx}$ is equal to:

$$\frac{dy}{dx} = (-3) \cdot (x^{-4}) - 2(-\sin 2x)$$
$$\frac{dy}{dx} = \frac{-3}{x^4} + 2\sin 2x$$

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10. Given the equation $y = 7\sin x - 24\cos x$.

(a) Putting in the form :

$$k\sin(x-a) \text{ where } k>0 \text{ and } 0 \le a \le \frac{\pi}{2}$$

We get:

$$k = \sqrt{7^2 + 24^2} = 25$$

$$\tan a = \frac{24}{7} \quad a = 1.29 rads$$

Hence we have

$$y = 7\sin x - 24\cos x = 25\sin(x - 1.29)$$

(b) To find x-coordinate in the interval $0 \le x \le \pi$ of the point on the curve where the gradient is 1, we have:

$$y = 25\sin(x-1.29)$$

$$\frac{dy}{dx} = 25\cos(x-1.29) = 1$$

$$25\cos(x-1.29) = 1$$

$$x = \cos^{-1}\left(\frac{1}{25}\right) + 1.29$$

$$x = 2.82 rads \quad \text{as required.}$$

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11. Given the claim that the wooden wheel is over 1000 years old. The amount of carbon in wooden wheel is 88%. To test claim we do the following.

$$A(t) = A_0 e^{-0.000124t}$$

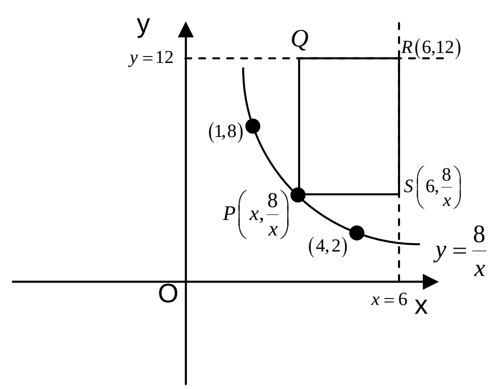
$$\frac{A(t)}{A_0} = 0.88 = e^{-0.000124t}$$

$$\ln(0.88) = -0.000124t$$

$$t = \frac{\ln(0.88)}{-0.000124} = 1031\ years$$

Therefore claim is true

12. Given the diagram, PQRS is a rectangle and is bounded by the lines x = 6 and x = 12, P lies on the curve $y = \frac{8}{x}$ between (1, 8) and (4, 2) and R is the point (6, 12).



(a) Length PS and RS are given by:

$$PS = \sqrt{(6-x)^2 + \left(\frac{8}{x} - \frac{8}{x}\right)^2} = (6-x)$$

$$RS = \sqrt{(6-6)^2 + \left(12 - \frac{8}{x}\right)^2} = \left(12 - \frac{8}{x}\right)$$

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12. (ii) Area of the rectangle is given by:

$$Area = length \times breadth$$
$$= (6 - x) \times \left(12 - \frac{8}{x}\right)$$
$$= 72 - \frac{48}{x} - 12x + 8$$
$$= 80 - 12x - \frac{48}{x}$$

(c) To find the greatest and least possible values for the area A and the corresponding values for x for which they occur, we have:

$$A = 80 - 12x - \frac{48}{x}$$
$$\frac{dA}{dx} = -12 + \frac{48}{x^2}$$

Greatest and least values occur at

$$\frac{dy}{dx} = -12 + \frac{48}{x^2} = 0$$
$$x^2 = \frac{48}{12}$$
$$x = \pm 2$$

From the diagram -2 is impossible.
Hence $x = 2$

Nature table shows x is a maximum:

1	2	4
+	0	-

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12. Maximum area A is:

$$A = 80 - 12(2) - \frac{48}{2} = 32 \text{ square units}$$

Minimum value will occur at one of the end points. Testing we get:

$$x = 1 \quad A = 80 - 12(1) - \frac{48}{1} = 20 \text{ square units}$$

$$x = 4 \quad A = 80 - 12(4) - \frac{48}{4} = 20 \text{ square units}$$

Least value of area occurs at x = 1 and x = 4.