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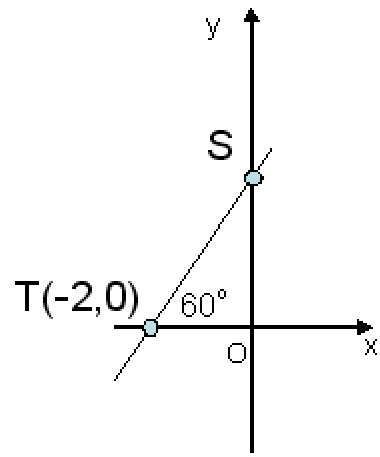
1. Given T(-2,0) and the diagram we can deduce

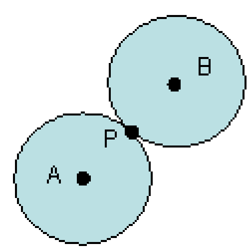


$$m = \tan(\theta^o)$$
$$m = \tan(60^o) = \sqrt{3}$$

Hence equation is

$$y - b = m(x - a)$$
$$y - 0 = \sqrt{3}[x - (-2)]$$
$$y = \sqrt{3}x + 2\sqrt{3}$$

2 Give the two congruent circles in the diagram and the equations:



$$x^2 + y^2 + 6x + 4y - 12 = 0$$
$$x^2 + y^2 - 6x - 12y + 20 = 0$$

(a) To find P we first find the centre of each circle

$x^2 + y^2 + 6x + 4y - 12 = 0$ Centre $(-g, -f) = (-3, -2)$

$x^2 + y^2 - 6x - 12y + 20 = 0$ Centre $(-g, -f) = (3, 6)$

Since circles are congruent P lies half-way between the centres.

$$P = \left(\frac{-3+3}{2}, \frac{-2+6}{2}\right) = (0, 2)$$

(b) Finding the length of AB we have.

$$\text{Distance\_formula} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$AB = \sqrt{[3 - (-3)]^2 + [6 - (-2)]^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

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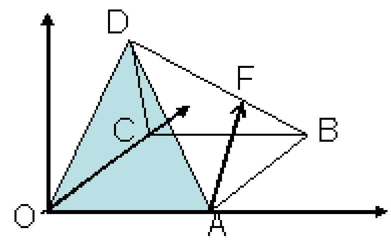

3. Given the pyramid and

$A = (12, 0, 0) \quad B = (12, 6, 0) \quad D = (6, 3, 9)$

F divides DB in the ratio 2:1

(a) First, find BF

$$BF = \frac{1}{3}BD = \frac{1}{3}\left[\begin{pmatrix}6\\3\\9\end{pmatrix} - \begin{pmatrix}12\\6\\0\end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix}-6\\-3\\9\end{pmatrix} = \begin{pmatrix}-2\\-1\\3\end{pmatrix}$$

$$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BF} = \begin{pmatrix}12\\0\\0\end{pmatrix} + \left[\begin{pmatrix}12\\6\\0\end{pmatrix} - \begin{pmatrix}12\\0\\0\end{pmatrix}\right] + \begin{pmatrix}-2\\-1\\3\end{pmatrix} = \begin{pmatrix}10\\5\\3\end{pmatrix}$$

Hence $F = (10, 5, 3)$

(b) AF in component form is

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF} = \left[\begin{pmatrix}12\\6\\0\end{pmatrix} - \begin{pmatrix}12\\0\\0\end{pmatrix}\right] + \begin{pmatrix}-2\\-1\\3\end{pmatrix} = \begin{pmatrix}-2\\5\\3\end{pmatrix}$$

4. Given the functions $f(x) = 3x - 1 \quad g(x) = x^2 + 7$

(a) Finding h(x) we have $h(x) = g(f(x)) = (3x - 1)^2 + 7$

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(b) (i) Minimum turning point for y = h(x) happens when

$(3x-1)^2 = 0 \qquad 3x-1=0 \qquad 3x=1 \quad x=\frac{1}{3} \qquad\qquad y=7$

Hence coordinates are $\left(\frac{1}{3}, 7\right)$

(b) (ii) Range of the function h is $7 \le h$ (The values that the y coordinate can take)

5. Differentiating the function: $y = (1+2\sin(x))^4$

$$\frac{d}{dx}y = 4\cdot(1+2\sin(x))^3\cdot(2\cos(x)) = 8\cos(x)(1+2\sin(x))^3$$

6. Given: $U_{n+1}=kU_n+5$

(a) For a sequence with the limit of 4 we require k to be

$$4 = k\cdot 4 + 5$$

Rearranging we get $k = \frac{4-5}{4} = \frac{-1}{4}$

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(b) Given a sequence satisfies the recurrence relation

$$U_{n+1} = mU_n + 5 \qquad U_0 = 3$$

(i) Expressing U_1 and U_2 in terms of (m) we have

$U_1 = mU_0 + 5 = 3m + 5$

$U_2 = m(3 \cdot m + 5) + 5 = 3 \cdot m^2 + 5m + 5$

(ii) The value of (m) which produces no limit for $U_2 = 7$

We have

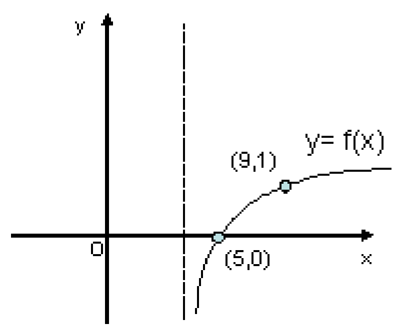
$3 \cdot m^2 + 5m + 5 = 7 \qquad 3 \cdot m^2 + 5m - 2 = 0 \qquad (3m - 1)(m + 2) = 0$

$m = \frac{1}{3}$ and $m = -2$

For a limit the magnitude of (m) must be less than 1

Hence value for no limit is $m = -2$

7. Given the function: $f(x) = \log_b(x - a)$



(a) From the graph we have

$a = 4$

$1 = \log_b(9 - 4) \quad b = 5$

Since $\log_5(5) = 1$

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(b) The domain of f is: $4 < x$

8. Given: $f(x) = 2x^3 - 7x^2 + 9$

(a) If (x-3) is a factor then by synthetic division we should have no remainder.

3	2	-7	0	9
		6	-3	-9
	2	-1	-3	0

Hence we have
$(x-3)(2x^2 - x - 3)$

Factorising fully we get $(x-3)(2x-3)(x+1)$

(b) For coordinates at the x-axis we have y = 0.

$(x-3)(2x-3)(x+1) = 0$ $\quad x = 3 \quad x = \frac{3}{2} \quad x = -1$

Coordinates are $(3, 0)$ $\left(\frac{3}{2}, 0\right)$ $(-1, 0)$

For coordinates at the y-axis we have x = 0.

$(0-3)(2\cdot 0 - 3)(0+1) = 9$ Coordinates are $(0, 9)$

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(c) To find greatest and least value between $-2 \le x \le 2$

We differentiate the function first to find the stationary points and then equate to 0.

$f(x) = 2x^3 - 7x^2 + 9$ $\quad\quad \frac{d}{dx}f = (6x^2 - 14x) = 2x(3x - 7) = 0$

$x = 0$ and $x = \frac{7}{3}$ (Note this value is outside range)

We can either construct a nature table or differentiate again to find nature of stationary points.

$\frac{d}{dx}(6x^2 - 14x) = 12x - 14$ $\quad\quad$ For $x = 0$ $\quad\quad 12 \cdot 0 - 14 = -2$

Hence $x = 0$ is maximum pt

Greatest value in the interval is: $\quad f(0) = 2 \times 0^3 - 7 \times 0^2 + 9 = 9$

Least value occurs at either x =-2 or x = 2 :

x = -2 $\quad\quad$ $f(-2) = 2(-2)^3 - 7(-2)^2 + 9 = -35$

x = 2 $\quad\quad$ $f(2) = 2(2)^3 - 7(2)^2 + 9 = -3$

So least value occurs at x = -2 and is -35

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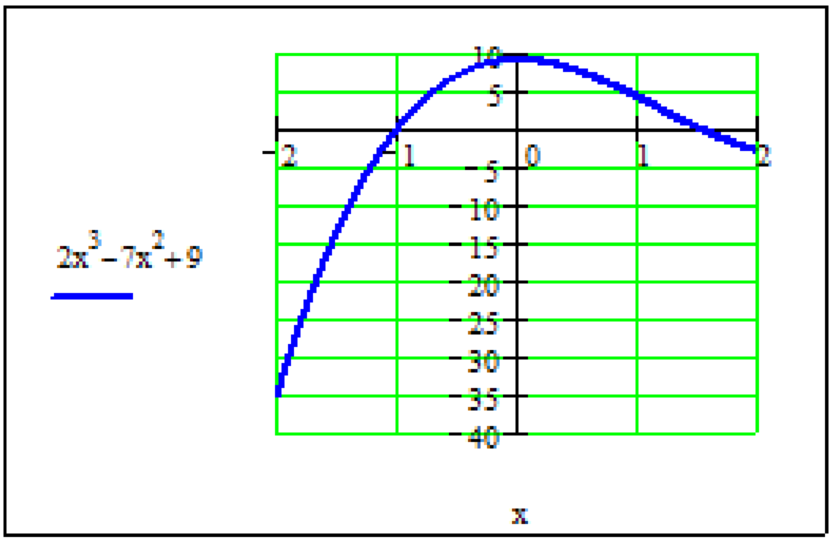
Sketch shown below



9. Given: $\cos 2x = \frac{7}{25}$ $\quad 0 < x < \frac{\pi}{2}$ $\quad$ Note we are only interested in values in the first quadrant.

To find $\cos x$ we use trig identity $\quad \cos 2x = 2\cos^2 x - 1 = \frac{7}{25}$

$$2\cos^2 x = \frac{7}{25} + 1 = \frac{32}{25} \qquad \cos^2 x = \frac{32}{50} = \frac{16}{25} \qquad \cos x = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

To find $\sin x$ we use trig identity $\quad \cos 2x = 1 - 2\sin^2 x = \frac{7}{25}$

$$2\sin^2 x = 1 - \frac{7}{25} = \frac{18}{25} \qquad \sin^2 x = \frac{18}{50} = \frac{9}{25} \qquad \sin x = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

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10. Given: $\sin(x) - \sqrt{3}\cos(x)$

(a) To put into the format $y = k\cdot\sin(x - a)$ $\quad k > 0 \quad 0 \le a \le 2\pi$

Comparing with the trig. identity

$$k\cdot\sin(x-a) = k\cdot\sin(x)\cdot\cos(a) - k\cdot\cos(x)\cdot\sin(a)$$
$$\sin(x) - \sqrt{3}\cos(x) = k\cdot\sin(x)\cdot\cos(a) - k\cdot\cos(x)\cdot\sin(a)$$

We get $\quad k\cdot\cos(a) = 1 \quad k\cdot\sin(a) = \sqrt{3}$

Squaring each side and adding we get

$$(k\cdot\cos(a))^2 + (k\cdot\sin(a))^2 = 1^2 + \left(\sqrt{3}\right)^2$$

$$k^2\cdot\left(\cos(a)^2 + \sin(a)^2\right) = 1^2 + \left(\sqrt{3}\right)^2$$

$$k^2 = 1^2 + \left(\sqrt{3}\right)^2$$
$$k = \sqrt{4} = 2$$

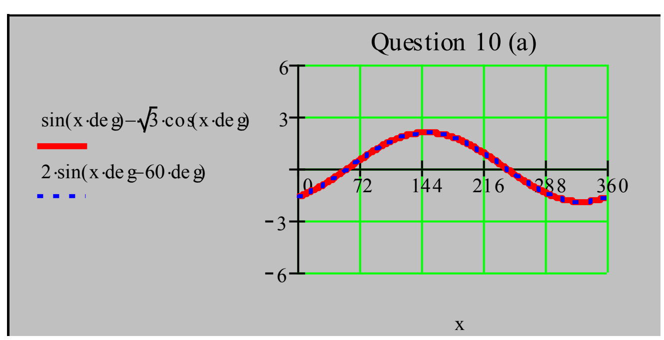
dividing both sides we get

$$\frac{k\cdot\sin(a)}{k\cdot\cos(a)} = \tan(a) = \frac{\sqrt{3}}{1}$$

$$a^o = \tan^{-1}\left(\sqrt{3}\right) = 60^o = \frac{\pi}{3}$$

Hence we have

$$\boxed{y = 2\cdot\sin\left(x - 60^o\right)}$$



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(b) To sketch $y = 3 + \left(\sin(x) - \sqrt{3}\cos(x)\right) \quad 0 \le x \le 2\pi$

We use part (a) to help us sketch function

The function can be rewritten as

$$y = 3 + \left(\sin(x) - \sqrt{3}\cos(x)\right) = 3 + 2\cdot\sin\left(x - 60^o\right)$$

We can now use the steps for sketch the graph using the basic sine graph.

-60^o Shifts basic sine graph to the right by 60^o
2 Stretch's the graph in the y-axis direction by a factor of 2
3 Moves graph up by a factor of 3

Cuts x-axis at y = 0, since there is no solution to

$x = \sin^{-1}\left(\frac{-3}{2}\right) + 60^0$ Graph does not cut x-axis

Max value occurs at $\left(150^o, 5\right)$ $\left(90^o + 60^0, 3 + 2\right)$

In radians $\left(\frac{5}{6}\pi, 5\right)$

Mini value occurs at $\left(330^o, 1\right)$ $\left(270^o + 60^0, 3 - 2\right)$

In radians $\left(\frac{11}{6}\pi, 1\right)$

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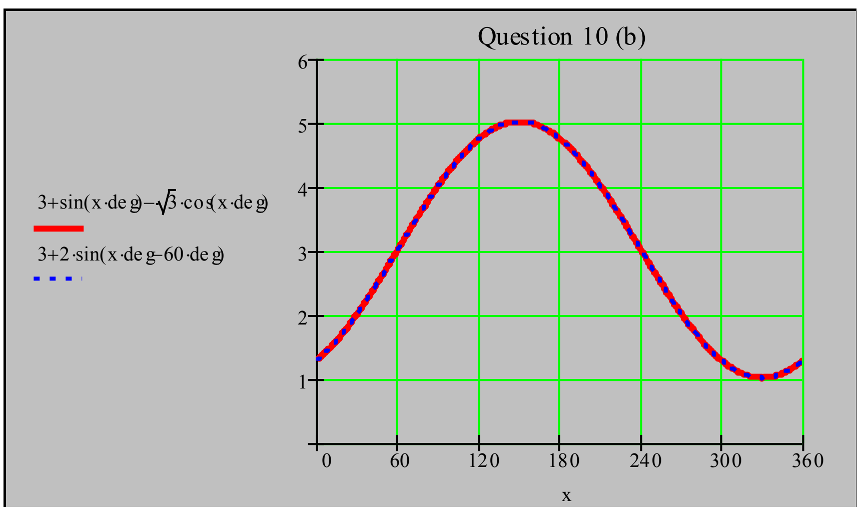
Cuts y-axis at x = 0

$$y = 3 + 2\cdot\sin\left(0 - 60^o\right) = 3 + \frac{\sqrt{3}}{2} = 1.3$$

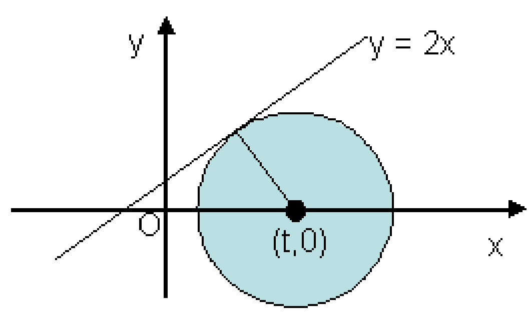
At $x = 360^o$ $y = 1.3$

Sketch is shown below



Q11. (a) Given the diagram and centre (t,0) $t > 0$

radius = 2



Equation of the circle is

$$(x - t)^2 + y^2 = 4$$

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(b) Exact value of t such that line is a tangent to the circle is:

$$(x-t)^2 + y^2 = 4$$

sub in y = 2x $\quad (x-t)^2 + (2x)^2 = 4$

expanding $\quad x^2 - 2xt + t^2 + 4x^2 = 4$

rearranging $\quad 5x^2 - 2tx + t^2 - 4 = 0$

Using the discriminant $\quad b^2 - 4ac = 0$

$$(-2t)^2 - 4(5)\left(t^2 - 4\right) = 0$$

$$4t^2 - 20t^2 + 80 = 0$$

$$-16t^2 = -80$$

$$t^2 = \frac{-80}{-16} = 5$$

$$t = \sqrt{5}, -\sqrt{5}$$

Since t>0 we pick the positive solution

$$t = \sqrt{5}$$