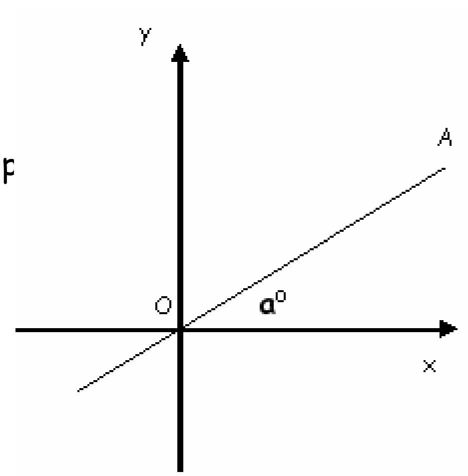
Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

1. Given the diagram and equation OA:

$$x - 2y = 0$$



(a) To find the angle a we use the p

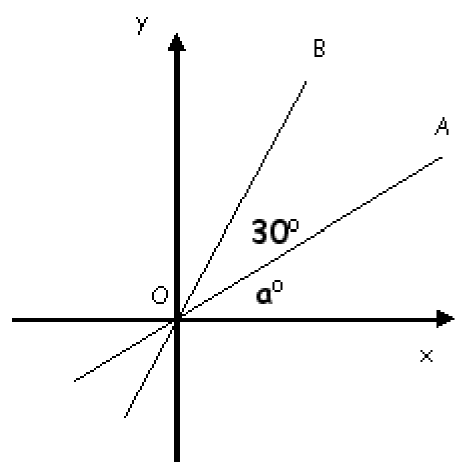
$m = \tan(a)^o$ $\quad$ $m = \text{gradient}$

$x - 2y = 0$

$a^o = \tan^{-1}\left(\frac{1}{2}\right)$ $\quad$ $y = \frac{1}{2}\cdot x$

$a^o = 26.57^o$

(b) To find the gradient of OB we have:-



$\text{AngleBOX} = 30^o + 26.57^o = 56.57^o$

Hence gradient of OB is

$m = \tan(56.57)^o = 1.5$

Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

2. Given: $P = (1, 3, -1) \qquad Q = (2, 0, 1) \qquad R = (-3, 1, 2)$

(a) In component form we have

$$\overrightarrow{QP} = p - q = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = -i + 3j - 2k$$

$$\overrightarrow{QR} = r - q = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} = -5i + j + k$$

(b) To find the angle PQR we have

$$a = -i + 3j - 2k \quad b = -5i + j + k$$

$$|a| = \sqrt{\left[(-1)^2 + 3^2 + (-2)^2\right]} = \sqrt{14} \qquad |b| = \sqrt{\left[(-5)^2 + 1^2 + 1^2\right]} = \sqrt{27}$$

$$a \cdot b = (-1)(-5) + (3 \cdot 1) + (-2) \cdot (1) = 6$$

Using the formula

$$a \cdot b = |a| \cdot |b| \cos\theta \qquad \cos(\theta^{o}) = \frac{a \cdot b}{|a| \cdot |b|} = \frac{6}{\sqrt{14} \cdot \sqrt{27}} = \frac{6}{\sqrt{36} \cdot \sqrt{\frac{21}{2}}} = \sqrt{\frac{2}{21}}$$

$$(\theta^{o}) = \cos^{-1}\left(\sqrt{\frac{2}{21}}\right) = 72^{o}$$

Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

3. Given $2x^2 + p \cdot x - 3 = 0$

If the roots are real for all p then the following is true.

$b^2 - 4 \cdot a \cdot c \geq 0$ $\quad p^2 - 4 \cdot 2 \cdot (-3) \geq 0$ $\quad p^2 + 24 \geq 0$

Since p^2 is always positive and 24 is always positive then

$p^2 + 24 \geq 0$ Hence roots are always real for any values p.

4. Given $U_{n+1} = k \cdot U_n + 3$

(a) For the sequence to have a limit we must have $-1 < k < 1$

(b) If the limit of the sequence is 5 we have

$L = k \cdot L + 3$

$L = 5$

$k = \frac{L - 3}{L} = \frac{5 - 3}{5} = \frac{2}{5}$

Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

5. Given the equation and point:

$y = 6 \cdot x^2 - x^3 \qquad P(x, y)$

(a) To find x for a gradient value of 12 at the point P, we have:

$$\frac{d}{dx}y = 12 \cdot x - 3 \cdot x^2 = 12$$

$$3x^2 - 12x + 12 = 0$$

$$x^2 - 4x + 4 = 0 \qquad (x-2)^2 = 0 \qquad x = 2$$

(b) To find the equation of the tangent at P we have

$$m = 12$$

For $x = 2 \qquad y = 6 \cdot \left(2^2\right) - 2^3 = 16 \qquad P(2, 16)$

Hence equation is

$$y - b = m(x - a)$$

$$y - 16 = 12(x - 2)$$

$$y = 12x - 8$$

Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

6. Given: $3\cos(x) + 5\sin(x)$

(a) To put into the format

$k\cdot\cos\left(x - a^{o}\right) \quad k > 0 \quad 0^{o} \le a \le 90^{o}$

Comparing with the trigonometry identity

$k\cdot\cos(x - a) = k\cdot\cos(x)\cdot\cos(a) + k\cdot\sin(x)\cdot\cos(a)$

$3\cdot\cos(x) + 5\cdot\sin(x) = k\cdot\cos(x)\cdot\cos(a) + k\cdot\sin(x)\cdot\sin(a)$

We get

$k\cdot\cos(a) = 3 \quad k\cdot\sin(a) = 5$

Squaring each side and added we get

$(k\cdot\cos(a))^2 + (k\cdot\sin(a))^2 = 3^2 + 5^2$

$k^2\cdot\left(\cos(a)^2 + \sin(a)^2\right) = 3^2 + 5^2$

$k = \sqrt{34}$

dividing both sides we get

$$\frac{k\cdot\sin(a)}{k\cdot\cos(a)} = \tan(a) = \frac{5}{3}$$

$k^2 = 3^2 + 5^2$

$a^{o} = \tan^{-1}\left(\frac{5}{3}\right) = 59^{o}$

Hence we have $y = \sqrt{34}\cdot\cos\left(x - 59^{o}\right)$

Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

(b) Solving $3\cos(x) + 5\sin(x) = 4 \qquad 0^o \le x \le 90^o$

We have

$$3\cdot\cos(x) + 5\cdot\sin(x) = \sqrt{34}\cdot\cos\left(x - 59^o\right) = 4$$

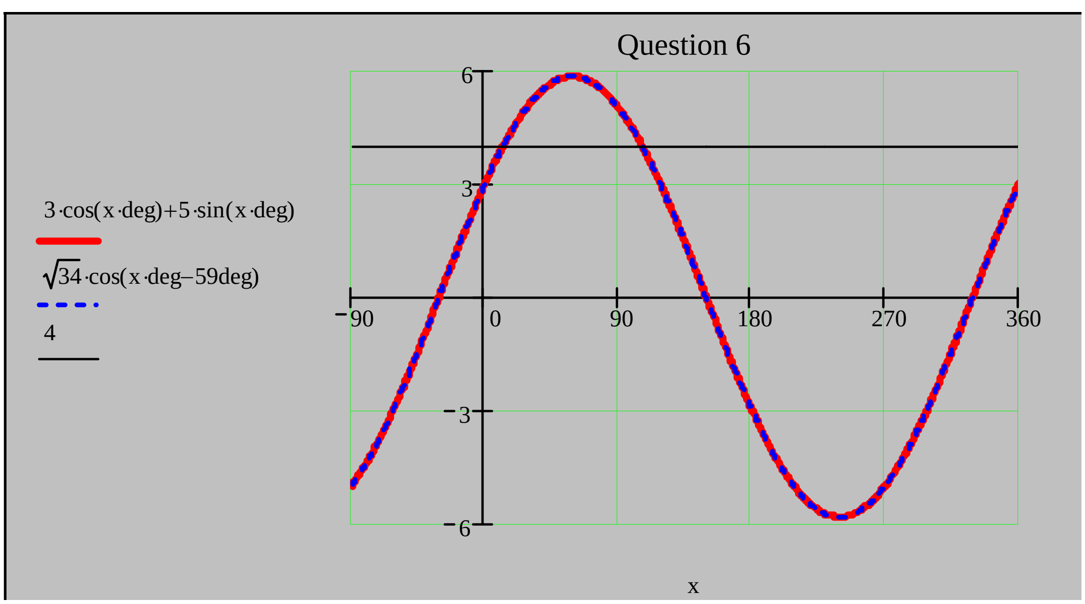
$$\cos\left(x - 59^o\right) = \frac{4}{\sqrt{34}}$$

$$\left(x - 59^o\right) = \cos^{-1}\left(\frac{4}{\sqrt{34}}\right) = 46.7^o \quad \text{and} \quad 313.3^o$$

$$x^o = 46.7^o + 59^o = 105.7^o \quad \text{and} \quad 313.3^o + 59^o = 372.3^o = 12.3^o$$

Since we are only concerned with the limit

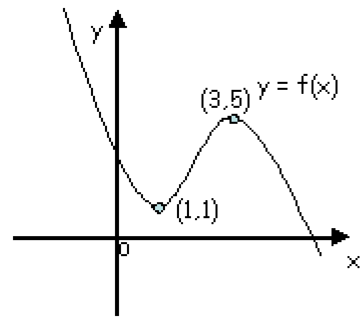
$$0^o \le x \le 90^o \qquad x^o = 12.3^o$$



Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

7. Given the cubic:



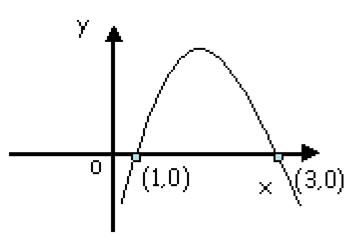
To sketch $y = \frac{d}{dx}f(x)$

First note that if we differentiate a cubic we get a quadratic.

Secondly, the coefficient of the cubic must be negative since for large values of x, y is large and negative and therefore the derivative function is a negative quadratic.

Thirdly, the points (1, 1) and (3, 5) are turning points and hence the derivative function at the points x = 1 and x = 3 must be 0.

Finally using the symmetry properties of the quadratic function we have:

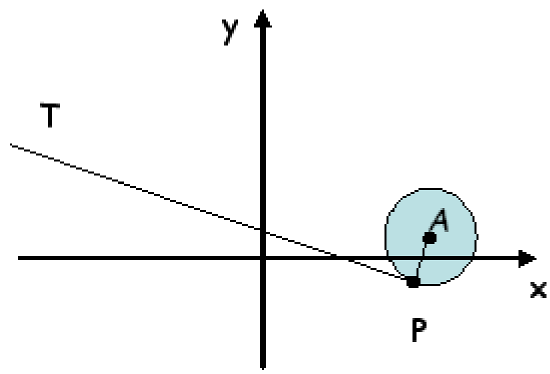


Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

8. Given: $x^2 + y^2 - 12x - 2y + 32 = 0$ $\quad P(5,-1)$

And the tangent PT



(a) To find the equation of the tangent we have:

Centre of circle is

$A(-g,-f) = (6,1)$

Gradient AP is given by

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-1)}{6-5} = 2$$

Since PT is a tangent we can use the formula

$m_{AP} \cdot m_{PT} = -1 \qquad m_{PT} \cdot 2 = -1$

$$m_{PT} = \frac{-1}{2}$$

Hence equation is

$y - b = m(x-a) \qquad m = \frac{-1}{2} \qquad P(5,-1)$

$$y - (-1) = \frac{-1}{2} \cdot (x-5)$$

$$2y + 2 = -x + 5$$

$$x + 2y = 3$$

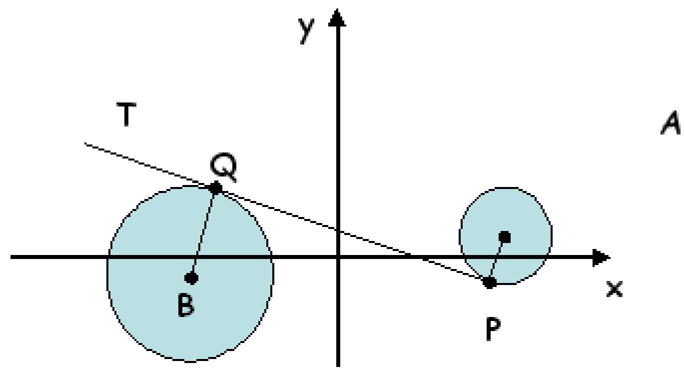
Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

(b) Given:

Big circle equation is

$x^2 + y^2 + 10x + 2y + 6 = 0$



If PT a tangent then if we substitute the equation for PT into the big circle equation then it should only have one real root.

$x = 3 - 2y$

$(3 - 2y)^2 + y^2 + 10(3 - 2y) + 2y + 6 = 0$

$9 - 12y + 4y^2 + y^2 + 30 - 20y + 2y + 6 = 0$

$5y^2 - 30y + 45 = 0$

$y^2 - 6y + 9 = 0 \qquad (y - 3)^2 = 0 \qquad y = 3$

Hence PT is a tangent to the big circle.

Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

(c) The y coordinate of Q is 3, x coordinate is given by:

x = 3 - 2y = 3 - 2(3) = -3

Hence Q has coordinates (-3, 3)

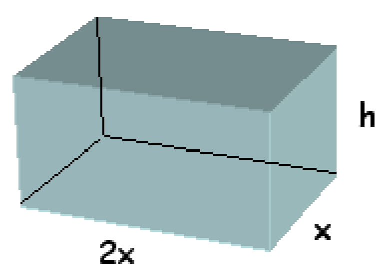
Length of Q is given by

$$L_{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[5 - (-3)]^2 + (-1 - 3)^2}$$

$$\sqrt{(8)^2 + (-4)^2}$$

$$\sqrt{80} \qquad \sqrt{16}\cdot\sqrt{5} \qquad 4\sqrt{5}$$

9. Given sketch and inner surface area of 12 units.



(a) The volume is given by:

$$\text{Volume} = \text{length}\cdot\text{breadth}\cdot\text{height}$$

$$V(x) = 2\cdot x\cdot x\cdot h = 2x^2\cdot h$$

But surface area is

$$A_s(x) = 2x\cdot h + 2x\cdot h + x\cdot h + x\cdot h + 2x\cdot x = 6\cdot x\cdot h + 2\cdot x^2$$

$$12 = 6\cdot x\cdot h + 2\cdot x^2 \qquad h = \frac{(6 - x^2)}{3\cdot x}$$

Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

Hence

$$V(x) = 2 \cdot x^2 \cdot \frac{(6 - x^2)}{3 \cdot x} \qquad V(x) = \frac{2}{3} \cdot x \cdot (6 - x^2)$$

(b) To find the maximum value for x for maximum volume we differentiate V(x) and equate to 0.

$$V(x) = \frac{2}{3} \cdot x \cdot (6 - x^2) = 4x - \frac{2}{3} \cdot x^3$$

$$\frac{d}{dx} V(x) = 4 - 2 \cdot x^2 = 0$$

Rearranging we get

$2 \cdot x^2 = 4$ $\qquad x^2 = 2$ $\qquad x = \sqrt{2}$ And $-\sqrt{2}$

We ignore the negative value since we cannot have a negative length.

Note: since the coefficient of x^2 is negative the value

$x = \sqrt{2}$ Gives the maximum volume

Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

10. Given: $A_t = A_o e^{-0.002 \cdot t}$ $t =$ years

$A_o =$ initial amount

(a) To find the initial value given:

$A_{1000} = 600$

We have

$600 = A_o \cdot e^{-0.002 \cdot 1000}$

$A_o = \dfrac{600}{e^2} = 4433$

(b) Half-life of the substance is given by:

$\dfrac{1}{2} = e^{-0.002 \cdot t}$

We have

$\log_e \cdot \left(\dfrac{1}{2}\right) = \left(\log_e(e)^{-0.002 \cdot t}\right)$

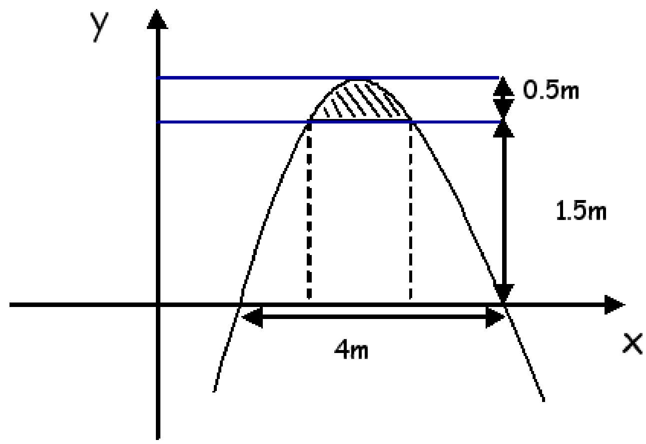
$-0.002t = \log_e\left(\dfrac{1}{2}\right)$

$t = \dfrac{\log_e\left(\dfrac{1}{2}\right)}{-0.002} = 346.6$

Higher Still Level Paper 2 2004

Created by
Graduate Bsc (Hons) MathsSci (Open) GIMA

11. Given the sketch and equation:



$$y = 2x - \frac{1}{2}\cdot x^2$$

To find the value of the shaded area we use the formula:

$$\int_a^b F(x) - G(x)\,dx \qquad F(x) = 2\cdot x - \frac{1}{2}\cdot x^2$$

$$G(x) = \frac{3}{2} = 1.5$$

Intersection of F(x) and G(x) is given by:

$$2\cdot x - \frac{1}{2}\cdot x^2 = \frac{3}{2}$$

$$4\cdot x - x^2 = 3$$

$$x^2 - 4\cdot x + 3 \qquad x^2 - 4\cdot x + 3 = (x-1)(x-3) \qquad x = 1 \quad \text{and} \quad x = 3$$

$$A = \int_1^3 \left(2\cdot x - \frac{1}{2}\cdot x^2\right) - \left(\frac{3}{2}\right) dx = \left(x^2 - \frac{1}{6}\cdot x^3 - \frac{3}{2}\cdot x\right) = \left(9 - \frac{27}{6} - \frac{9}{2}\right) - \left(1 - \frac{1}{6} - \frac{3}{2}\right)$$

$$A = \frac{2}{3}$$